

The Sweet 'Bees-ness' of Honeycomb



... and the Mathematics of Hexagonal Geometry

TEACHER ANSWER BOOKLET

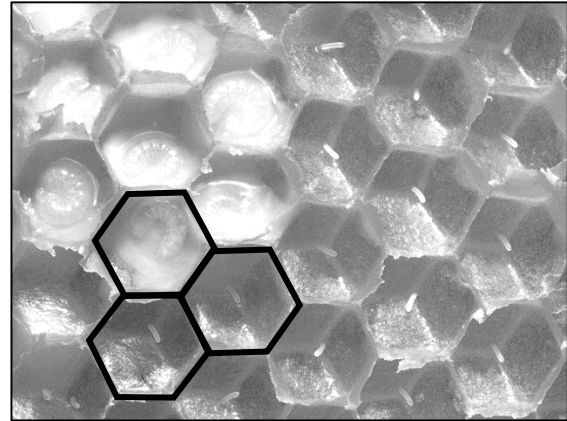
Image: Wentzel, L. (2014), 'Musical Animals: Concertina Beehive'. An idea for Ypsilanti's Honeybee Festival. Courtesy of 'Creative Commons'. Accessed from <https://www.flickr.com/photos/wentzelepsy/14507537589>. Image labelled for reuse.

The Sweet 'Bees-ness' of Honeycomb

A. Introduction – What is Honeycomb?

Mmmm, honey! Almost everyone loves this sweet, natural treat. Many people will also tell you that the best way to enjoy honey is straight from the 'comb' – the honeycomb, that is!

Have you ever taken a good hard look at a piece of honeycomb? What shape are the cells in a section of honey bee honeycomb? It's a **regular polygon** with **six sides**... yes, it's a **hexagon**!



- (i) What is a polygon? **A polygon is a two-dimensional enclosed shape with straight sides.**
- (ii) What does it mean for a shape to be 'regular' in geometry? **It means the shape has all its sides an equal length. (Regular shapes also have all angles equal).**

Of course, humans are interested in honeycomb because it contains honey, which we like to eat. However, bees create honeycomb to use them for a number of needs: to make honey out of nectar, to store honey, to store pollen, to lay eggs and to raise young bees (larvae) into adults¹. Honeycomb cells are truly 'multipurpose spaces'!

Honeycomb is constructed by bees out of a waxy substance called beeswax, which is made by young bees and takes a lot of energy for them to make. Remember, **energy** – the ability to do work – is like money for bees in that it is energy that allows them to collect food, build shelter and raise young – and so survive. So, things that use a lot of energy in the animal kingdom are said to be 'expensive'². This means that the more energy animals (and insects) surviving in the wild can conserve, the better!

B. Honeycomb Geometry


Each cell in a section of honeycomb has a **hexagonal** ('hexagon-like') opening and is in the shape of a type of **irregular polyhedron** called a **decahedra**³ (Phew! What a mouthful!). This means that the honeycomb cell has 10 flat sides – the prefix '*deca*' means '10' and the suffix '*-hedra*' means 'geometric figure'. Also, all the flat sides of honeycomb cell except for the hexagonal opening are **quadrilaterals**.

- (i) What is a quadrilateral? **A polygon (2 dimensional shape with straight sides) that has four sides.**
-
- (ii) In the space on the following page, draw an example of one type of quadrilateral and list its features or 'properties'.
(By the way...Is a square a quadrilateral? Is a quadrilateral always square? *You may wish to discuss this with your class and your teacher!*)
(Class discussion about how a square is a quadrilateral, but a quadrilateral is not always a square...)

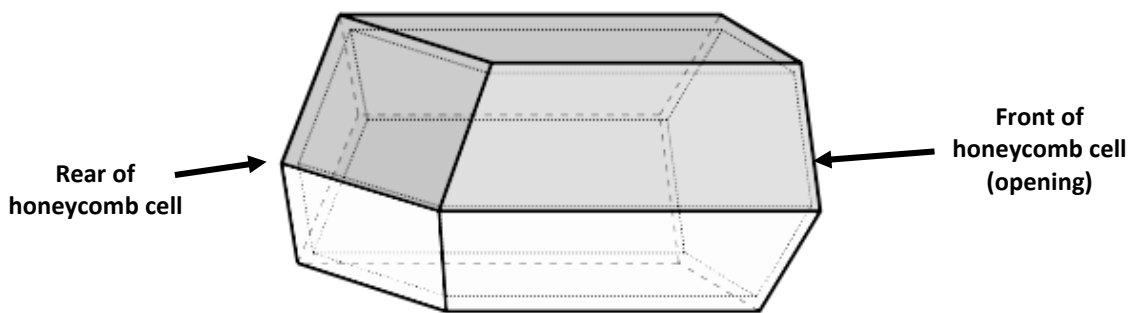
¹ Reichert, T. (2015), 'Why do we see hexagons so often in nature?' Online article on Quora.com (URL: <https://www.quora.com/Geometry/Why-do-we-see-hexagons-in-nature-so-often/answer/Tal-Reichert>) Accessed 20/11/2015.

² Ibid.

³ Frisch, Karl von (1974). *Animal Architecture*. New York: Harcourt Brace Jovanovich.

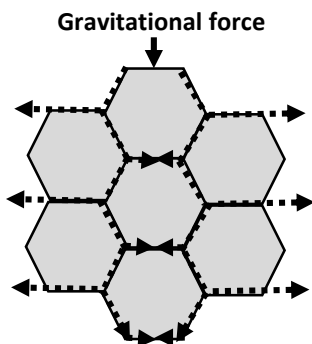
<p>Eg.</p> 	<p>Name of your quadrilateral: <u> Eg. Parallelogram </u></p> <p>Properties of a quadrilateral:</p> <ul style="list-style-type: none"> • <u> Four sides </u> • <u> Opposite sides parallel </u> • <u> Opposite angles equal </u>
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(iii) Can you identify the 9 **quadrilaterals** and the **hexagon** on the diagram of a honeycomb cell below⁴? Label them from 1 to 9 and then label the regular hexagon that forms the cell opening as number '10':



The closed ends of the honeycomb cells are a bit more complicated – they are **trihedral** (i.e., composed of three flat planes), ensuring that the back-to-back ends of honeycomb cells fit neatly against each other and that the overall shape of the honeycomb cell *minimizes surface area for a given volume*. This means that the inside of the cell gives the bees maximum space (for growing young or storing honey) using minimum building materials (ie, beeswax). Proving this fact is a complicated little piece of three-dimensional geometry – however, it was done years ago by a couple of mathematicians called Jan Brozek and Thomas Hales⁵.

This is important for bees because it means that if they build a structure from beeswax it needs to be as **energy inexpensive** as possible and so it should minimise the amount of beeswax used. At the same time, however, the structure needs to be physically strong, leak-proof, resistant to attack from predators and well-suited to the range of purposes for which it was built. Quite a difficult design brief for even the best of architects!



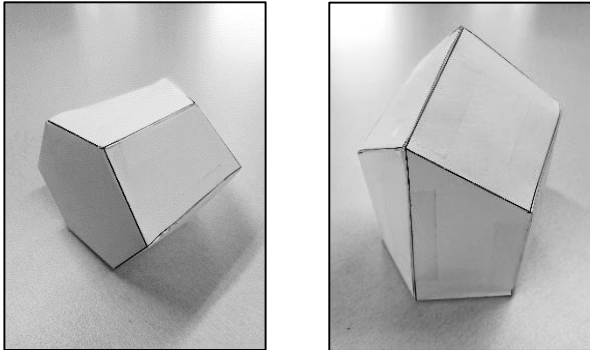
Indeed, **hexagonal structures are very strong**. Honeycombs are usually constructed vertically (eg, up a wall) and so they must be able to stand up firmly on their own and resist the downward pull of gravity. Hexagons work well because the shared walls of each cell ensure that the downward physical stresses on the structure are spread evenly throughout (see diagram). This makes them very resilient.

⁴ Diagram courtesy of Wikipedia, 'Honeycomb Geometry', URL: <https://en.wikipedia.org/wiki/Honeycomb>. Image labelled for reuse. Accessed 23/11/15. Labels added by author.

⁵ Ibid.

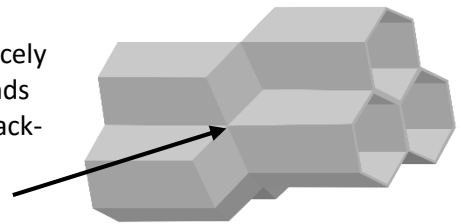
(iv) **Build your own honeycomb cell.** For this activity you will need a pair of scissors and the shape net diagram on the next page.

Your completed decahedron should look something like the one in the illustrations below:



Once you have finished making your honeycomb cell, get together with some friends and see if you can group your cells together to make a three-dimensional model of a honeycomb.

Do your cells fit nicely together at the ends (ie, back-end to back-end, in clustered groups) as well⁶?

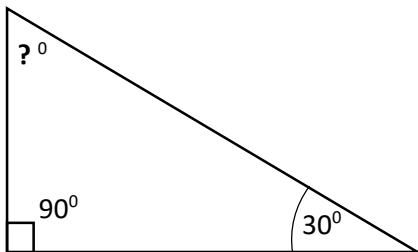


(v) **Angles inside a Regular Hexagon.**

Can you remember (or do you know) how many degrees there are if I add up the angles inside any triangle?

That's right! It's **180°**!

This means that once we know the measurement of any two angles inside a triangle, we can work out the third one – for example, in a right angled triangle we might have one angle of 90° (of course), and another angle of 30°:



Total degrees in a triangle = 180

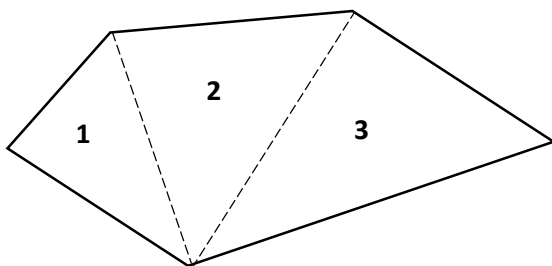
Total degrees in the two angles we know = 90 + 30
= 120

Therefore, the missing angle we don't know ('?') is calculated as:

$$\begin{aligned} ? &= 180 - 120 \\ &= 60^\circ. \end{aligned}$$

Knowing that all the angles in a triangle add to 180° is also useful for working out the number of angles in other types of polygons as well! All we need to do is work out how many triangles occur in a given polygon and then we multiply this number by 180° to get the total degrees in that shape.

Let's take the following **irregular pentagon** as an example (hmmm... why is it called an *irregular pentagon*?):



Here, we can count 1, 2, **3** triangles within the pentagonal shape simply by dividing the shape with diagonal lines that dissect and join each internal angle.

If there are 3 triangles within the shape, this means there is 180° x 3 = **540° in the pentagon shape**. In fact, this is true of *any* pentagon, whether regular or irregular.

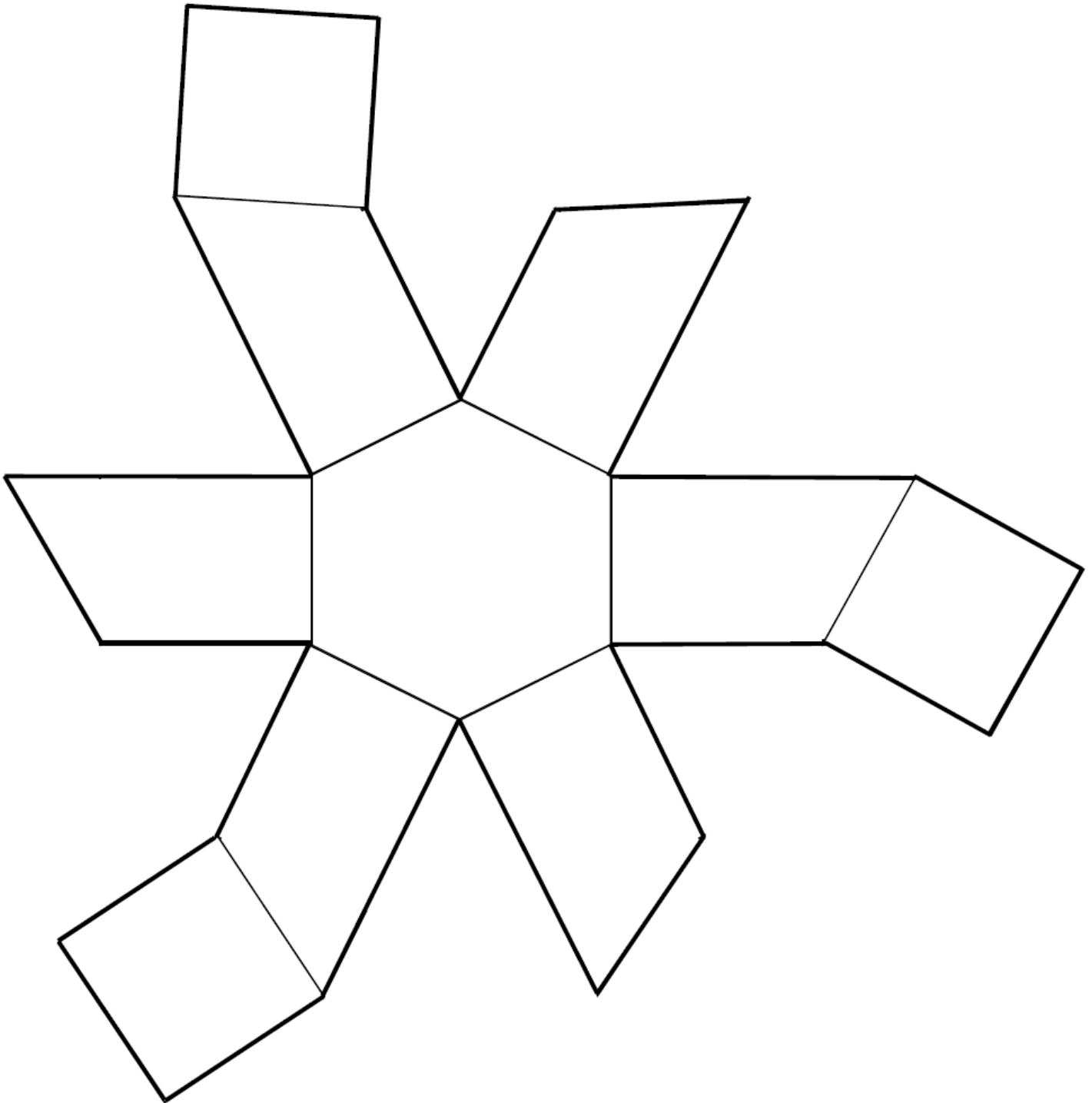
That is to say, if I measured and added up the angles of any pentagonal (5 sided) polygon, I would get a total of 540°!

⁶ Ibid.

Build your Own Honeycomb Cell – Decahedron Shape Net

Cut the shape below out around the darker outlines, then fold along the thinner internal lines of the shape net.

Grab some sticky tape and carefully stick the shape together along the edges. Hey, presto! You have your very own honeycomb cell!

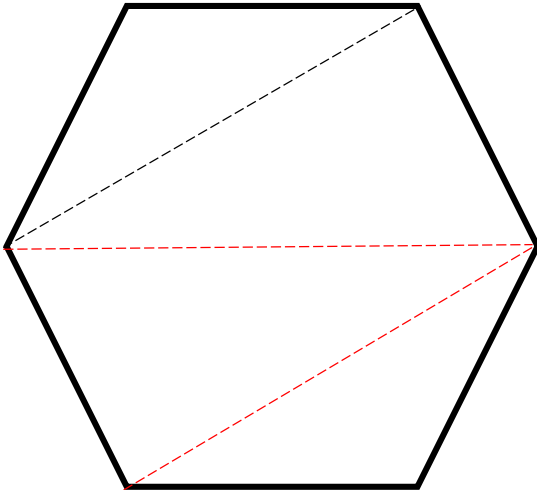


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(‘Angles inside a hexagon’ continued...)

Now we’re ready to solve a problem. **How many degrees in a hexagon, and without using a protractor to measure, can we find out the measurement of each angle in a regular hexagon?**

First, divide up the following regular hexagon into triangles by using your ruler and a pencil to draw diagonal lines that dissect the internal angles of the **regular hexagon** below. The first one has been done for you:



Count the number of triangles in the hexagon and multiply this by 180° to get the **total number of degrees inside a hexagon**: $4 \times 180 = 720^\circ$

Next count the number of angles there are inside the hexagon and write the answer here: 6

In any regular shape (ie, a shape where all the sides are the same length), **all the angles will also be exactly the same number of degrees**. For example, in an equilateral (regular) triangle (with 3 equal sides and therefore 3 equal angles), each angle = $(180^\circ \div 3) = 60^\circ$.

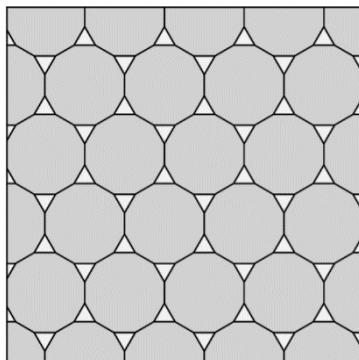
So, **each angle in a regular hexagon** = [total degrees in a hexagon] \div [number of angles in a hexagon]

$$= \frac{720}{6}$$

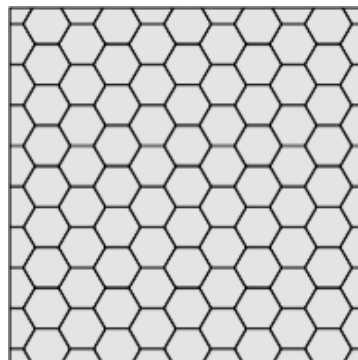
$$= 120^\circ$$

C. Terrific Tessellations

Honeycombs on beehives demonstrate a hexagonally **tessellating pattern**. This means that all the shapes in the pattern fit neatly next to one another with no gaps or ‘blank spaces’ between them. In particular, hexagons can form a **regular tessellation**, which means that we only need to repeat that same shape (that is, we don’t need additional shapes to ‘fill the gaps’) to form a tessellating pattern:



Non-regular Tessellating Pattern
(Note the ‘gaps’ between the shapes)

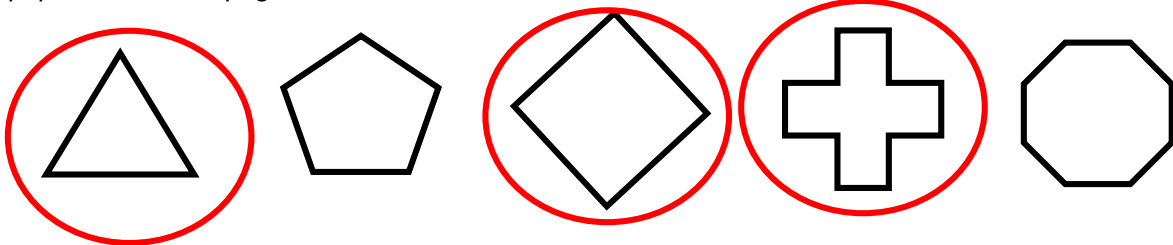


Regular Tessellating Pattern
(Same shape repeated – no gaps!)

Not all shapes can tessellate regularly – however, as you can see in the above pattern on the right hand side, hexagons tessellate very nicely indeed!

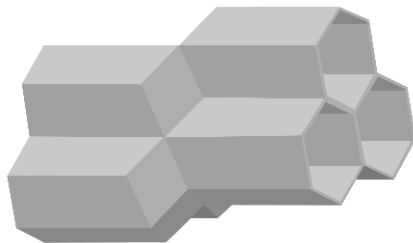
This is kind of a big deal for bees, because it means that all the cells in the honeycomb share each and every wall with other cells in the honeycomb. This is very efficient, minimising the amount of beeswax needed and thus saving the bees lots of energy!

- (i) Have a look at the regular shapes below. Which of these would enable a regular tessellation pattern? You may wish to explore this by trying to create your own tessellating patterns using these shapes using the grid paper on the next page.

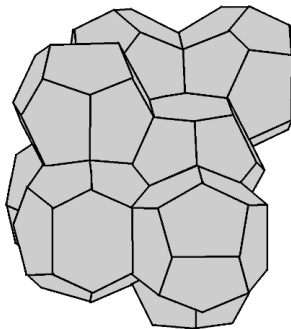


Shapes circled can form regular tessellation patterns

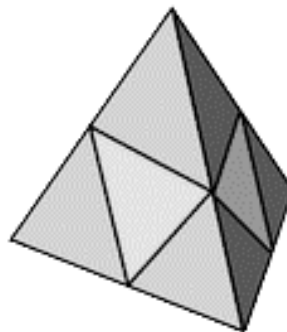
- (ii) Look again at the diagram showing the three-dimensional models of honeycomb cells interlocked together:



So, three dimensional objects can also tessellate! Here are some examples of other three dimensional tessellating patterns:



Tessellating Regular Pentagonal Dodecahedra



Simple Tessellating Triangular Pyramids

Use your favourite search engine to find out the **general term** used for three dimensional tessellations:

'Honeycombs'

Were you surprised at this mathematical term? Why do you think three-dimensionally tessellating objects have been given this name by mathematicians?

Probably because natural honeycombs tessellate in three dimensions!

D. Honeycomb Algebra

a. *Some background to using Algebra*

Algebra is a way of thinking mathematically about the structure of patterns and situations in the real world, not just a series of ‘maths rules’ that involve strange uses of letters of the alphabet!

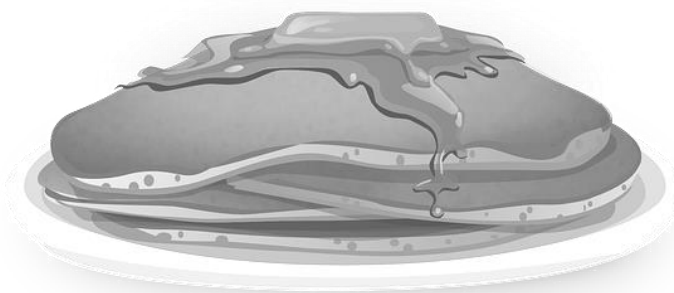
Algebra can help us predict what will happen to the results in a pattern or a mathematical situation when we change some of the ‘inputs’ or ‘variables’ in the pattern or system.

A useful way of thinking about this might be with a recipe. Consider the following

To make simple pancake batter for making one batch of 8 (good sized) pancakes you would need:

- 1 cup self-raising flour (let’s call this unit quantity of flour ‘ f ’, for ‘flour’);
- 1 cup whole milk (‘ m ’ for ‘milk’)
- 2 egg (‘ e ’ for ‘eggs’)
- 1 pinch of salt (‘ s ’ for ‘salt’).

Let’s also call a batch of 8 pancakes ‘ \mathcal{P} ’ (for pancakes!).



There are other recipes that use the same ingredients (‘variables’) – but in different amounts.

Using algebra allows us to vary the ingredients to get a different overall result of ‘output’.

Now, I’m going to describe my pancake batter recipe using algebra!

Here we go:

$$f + m + e + s = \mathcal{P}$$

This way of expressing the ‘formula’ for my recipe is called an **algebraic equation**.

This simply means that one unit (cup) of flour plus one unit (cup) of milk plus one unit of egg (2 eggs) plus one unit of salt (pinch) will equal batter for 1 batch of pancakes (8 pancakes).

But what if I'm feeding a hungry horde of 12 year olds for breakfast after a birthday party sleepover? I might need 3 times that many – say, 24 pancakes! This means I will need to multiply my recipe by 3, so:

$$3 \times (f + m + e + s) = 3P$$

Thinking about each of my ingredients in turn, we can rewrite the equation as follows:

$$3f + 3m + 3e + 3s = 3P$$

(So, 3 units of flour plus 3 units of milk plus... etc... equals batter for 3 batches, or 24 pancakes!)

Thus, algebra is a useful way of expressing in 'maths language' the way a pattern or system is made up of parts, and it can also help us see what happens to the outputs of a system or pattern (eg. number of pancakes) when we change the inputs in the system (eg. the ingredients for pancake batter).

- (i) Can you write the equation we would need if we wanted **56** pancakes? (Remember, $P = 8$ pancakes):

$$7f + 7m + 7e + 7s = 7P$$

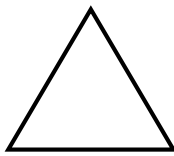
How many cups of flour would this need ($f = 1$ cup)? 7

How many eggs ($e = 2$ eggs)? 14

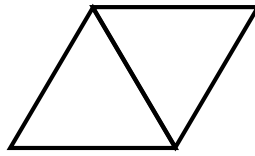
b. Algebra and Geometry

When we construct a geometric or shape pattern, we use the ‘inputs’ of lines and angles. Algebra can also tell us how many lines or angles we would need to use to generate a certain number of shapes in a given pattern.

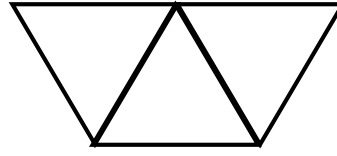
Have a close look at the linear (ie, in one row) tessellating triangular pattern below:



1 triangle
3 lines



2 triangles
5 lines



3 triangles
7 lines

In this pattern, as we add 1 new triangle, each new triangle shares a side with the previous triangle.

(ii) How many lines for 4 triangles? 9 How many for 6 triangles? 15

(iii) We’ll pop the relationship between triangles and lines into an ‘Input/Output’ table. Complete the table, as follows:

Number of Triangles in Pattern (<i>Outputs</i>)	Number of Lines (<i>Inputs</i>)
1	3
2	5
3	7
4	9
5	11
6	13
7	15

Now, the magic of algebra is that we can use it to help us to accurately predict how many inputs (lines) we would need to produce any given number of outputs (triangles).

Let’s first refer to the number of lines we need as the letter ‘*ℓ*’ (for ‘lines’).

We will also call the number of triangles in the pattern ‘*t*’ (for ‘triangles’).

Using trial and error, thinking about the relationship between lines and triangles (*ℓ* and *t*) in this pattern, we can see that the number of lines *ℓ* is equal to 2 times the number of triangles *t*, plus 1. We can write this as an **algebraic equation**:

$$\ell = (2 \times t) + 1$$

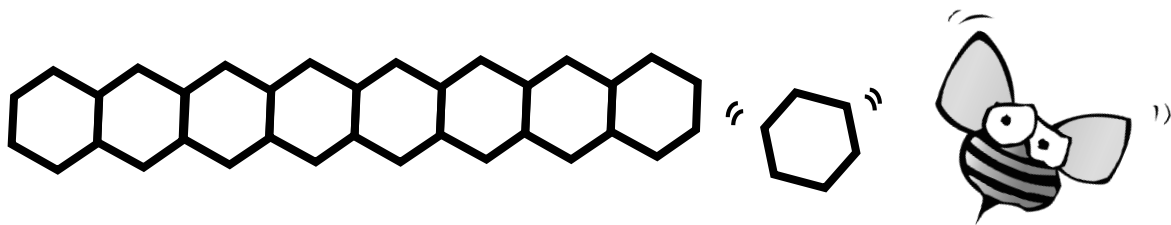
In most algebraic equations, we can be a bit 'lazy' and get rid of the multiplication symbol (\times), assuming that if two terms are right next to one another they are being multiplied. Also, applying our rules of 'order of operations', we see that we don't need the brackets, either. So, we can write the equation describing this pattern as follows:

$$l = 2t + 1$$

This little equation is actually very powerful as we can now use it to work out how many lines we will need to create a much larger number of triangles. If I want to make this pattern containing 200 triangles, for example:

$$\begin{aligned}
 l &= 2t + 1 \\
 \text{Therefore, } l &= (2 \times 200) + 1 \\
 &= \mathbf{401}
 \end{aligned}$$

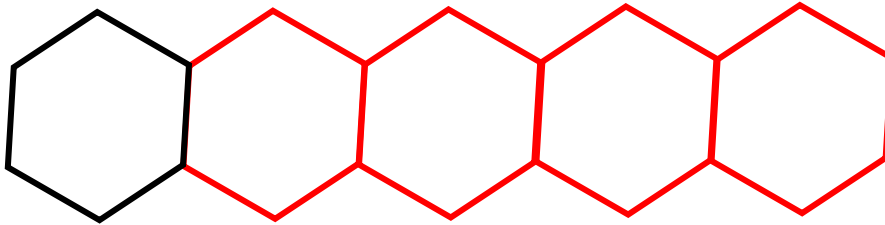
Therefore, I will need 401 lines to create a triangular tessellating pattern containing 200 triangles.



c. Hexagonal Algebra Patterns

Honeycombs are arranged in tessellating hexagonal patterns, arranged in linear rows.

- (iv) Using a ruler, draw in an additional four hexagons onto the hexagon below, so you have a single row of 5 hexagons:



- (v) Complete the missing values in the table for the relationship between the number of hexagons (h) in a linear pattern and the number of lines (l) drawn to construct the pattern:

DRAFT

Number of Hexagons (h)	Number of Lines in total (l)
1	6
2	11
3	16
4	21
5	26

- (vi) Look back over the material on page 11, using an algebraic equation to describe the relationship between lines and shapes in the triangle pattern. Also, look carefully at the table of values for the hexagon pattern above.

Using your own reasoning and some trial and error, can you write an algebraic equation that describes the relationship between the number of hexagons and the number of lines? The structure of the equation has been provided for you:

$$l = (\underline{5} \times h) + \underline{1}$$

ie, $l = 5h + 1$

Use your algebraic equation to calculate how many lines (l) would be needed to construct a linear pattern with 60 hexagons ($h = 60$). Be sure to show your working!

$$l = 5h + 1$$

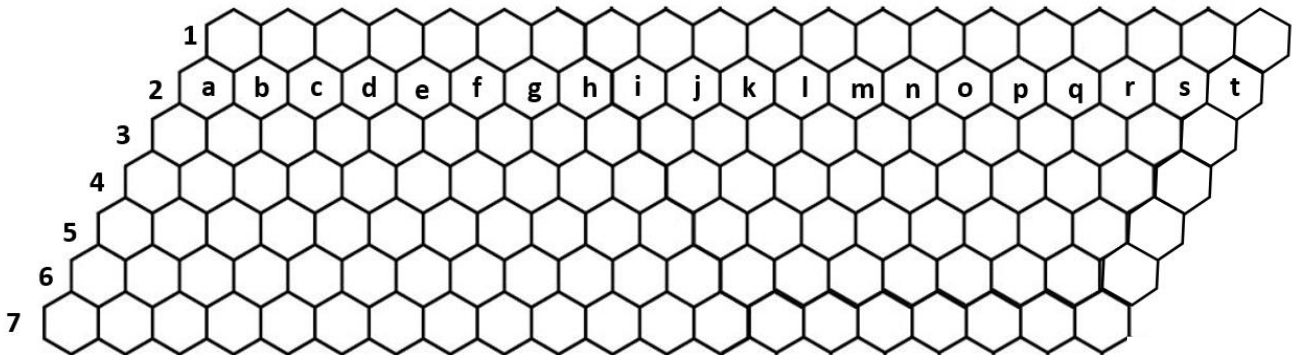
Therefore, $l = 5 \times 60 + 1$

$$= 301$$

E. Extension⁷: Algebraic description of Hexagonal Tessellation

Consider the following section of honeycomb containing 140 separate cell shapes.

There are 7 rows of exactly **20 cells** (labelled as ‘a’ through to ‘t’ on the diagram) **in each row**. We have labelled the **rows** 1 through to 7:



- (i) What if I were a (particularly clever) worker bee and I wanted to work out how many cell walls needed to be produced to construct this section of honeycomb?

In Row 1, our equation is as per the formula on the bottom of Page 12 above,

$$l_{(\text{Row } 1)} = \underline{5} n_{(\text{Row } 1)} + \underline{1} \quad \dots(1)$$

However, consider then adding Row 2. By adding the first cell (cell ‘2a’ in the diagram), we are adding 5 walls to produce a new cell. However, for cells 2b through to 2t, we need only add 3 walls to complete each new cell.

The same applies for each other row from 2 through to 7.

⁷ Suitable for a Year 7 or 8 Level Enrichment / Extension task

Thus, for Rows 2 to 7 (ie, 6 rows in total), the input/output table describing the mathematical relationship between the number of new lines (or new cell walls) ' \mathcal{L} ', and the number of new hexagons (or new cells) created ' \mathcal{H} ', for each row, is as follows:

Number of New Cells in each of Rows 2 – 7 (Outputs) ' \mathcal{H} '	Number of New Cell Walls Created in each of Rows 2-7 (Inputs) ' \mathcal{L} '
1	5
2	8
3	11
4	
5	

The algebraic equation for this relationship for each separate rom (ie, Rows 2 – 7 inclusive) is thus:

$$\mathcal{L}_{(\text{Rows 2 to 7})} = \underline{3} \mathcal{H}_{(\text{Rows 2 to 7})} + \underline{2} \quad \dots(2)$$

Now, in this configuration of honeycomb there are a total of 7 rows.

The first one (Row 1) follows the formula at (1) above, and the next 6 rows follow the formula at (2) above.

Thus, to calculate the total number of cell walls (\mathcal{L}), we would have the following algebraic expression:

$$\mathcal{L}_{(\text{Total})} = \mathcal{L}_{(\text{Row 1})} + \mathcal{L}_{(\text{Rows 2 to 7})}$$

Therefore,

$$\mathcal{L}_{(\text{Total})} = \underbrace{(\underline{5} \mathcal{H}_{(\text{Row 1})} + \underline{1})}_{\text{from (1) above}} + 6 \times \underbrace{(\underline{3} \mathcal{H}_{(\text{Rows 2 to 7})} + \underline{2})}_{\text{from (2) above}} \quad \dots(3)$$

We can express the number of rows from Row 2 to Row r inclusive as ' $r-1$ '. Thus, the formula to calculate the number of cells for Rows 2 through to Row r (whatever that might be) is:

$$l_{(\text{Rows 2 to } r)} = (r-1) \times [(\underline{3} \ h_{(\text{Rows 2 to } r)} + \underline{2})] \quad \dots(6)$$

To work out the number of cell walls needed altogether, I will need to add the total for Row 1 together with the total for Rows 2 to r .

Combining equations (5 and (6) above, we therefore have:

$$l_{(\text{Total})} = [(\underline{5} \ h_{(\text{Row 1})} + \underline{1})] + (r-1) \times [(\underline{3} \ h_{(\text{Rows 2 to } r)} + \underline{2})] \quad \dots(7)$$

Use Equation 7 to calculate how many cell walls would need to be produced to construct a section of honeycomb that has 60 rows, with 40 cells in each row. Show your working!

$$l_{(\text{Total})} = [(5 \times h) + 1] + (r-1) \times [(3 \ h_{(\text{Rows 2 to } r)} + 2)] \quad \dots(7)$$

l = Total number of cell walls to be produced

$$h = \underline{40}$$

$$r = \underline{60}$$

$$l_{(\text{Total})} = [(5 \times 40) + 1] + 59 \times [(3 \times 40 + 2)]$$

$$= 201 + (59 \times 122)$$

$$= 201 + 7\ 198$$

$$= 7\ 218$$

Therefore, the total number of cell walls needed to construct a section of honeycomb that has 60 rows with 40 cells in each row is: 7 218