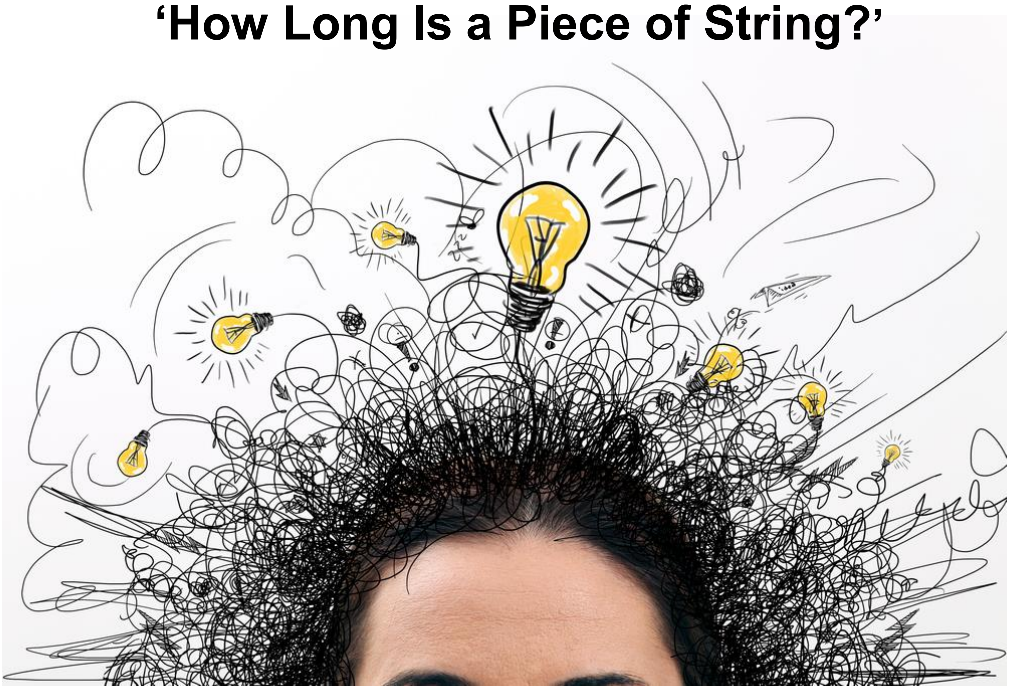
‘How Long Is a Piece of String?’



Using Open-ended Problem Solving to Deepen Mathematical Understanding

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Consider the following real-world mathematics problem:

> *“There’s nothing like a hot meat pie and chips (with an accompanying refreshment) at the footy!*
>
> *On Saturday night, my friend and I had $40 each to spend on food and drinks at the game. I had my son with me and my friend had his daughter (both of whom were under 18).*
>
> *Pies are $4.00 each and hot chips are $4.50 each. Water is $3.00 and soft drinks are $4.50. Cold beer is $9.70 a cup.*
>
> *What are some ways we could feed and refresh our group of four people over the two hours or so at the footy, within our $80 budget?”*

- What is it that might make this a *difficult* task for students in, say, grades 4 through 8?
- What is it that might make it *interesting* or engaging for students at these levels?
- How might you go about helping your students to solve this particular task? For example, what prompts or resources might you provide (if any)?
- How much time would you give your students to solve the problem?
- How many solutions might you expect them to find?

This article will focus on considering how open-ended problem solving is different from ‘regular’ maths questions or problems, and discuss the significant benefits of using this pedagogical approach to mathematics, for both students and for teachers.

Section 1: What is open-ended problem solving and how is it different?

(i) Problem solving in mathematics

The skill that we generally call 'problem solving' in mathematics (or any field of academic endeavour) is a complex one. Essentially, problem solving refers to the process of finding a solution to a specific mathematical situation, whereby that process is essentially 'non-linear', that is, there is rarely a direct or single correct pathway to find an answer. The problem-solver must first consider or formulate the nature of the work to be done, explore several possibilities, perhaps go back and forth in some process of trial and error and then check the solution against the original question or problem to ensure it 'fits'. Most problem solving allows several or numerous ways in which to find a solution, and many problems may indeed have more than one correct solution.

Mathematical problem solving can be considered as a cycle of cognitive activity, although the cycle often requires us to move backward and forward whilst maintaining a general sense of direction. There are many models of the problem solving cycle (Piggott, 2008). 'N-RICH Mathematics' suggests the following Problem Solving cycle (Handout 1), which explains how students / mathematicians might move back and forth between the stages of:

- Identifying / interpreting the nature of the problem;
- Representing the problem / situation mathematically;
- Analysing the problem using mathematical procedures ('operations') and reasoning;
- Interpreting results / outcomes, especially within the context of the original problem;
- Reflecting upon whether there might be better solutions or more efficient methods; and
- Communicating the results as appropriate.

The application of the problem-solving cycle is a high-order thinking skill and evidence suggests that few pupils utilise the problem-solving cycle effectively.

Australian educational researcher Anne Newman describes five steps students go through to answer any worded maths problem: Reading, Comprehending, Transforming, Processing and Encoding / Applying. In her research, Newman found that 50% of students' errors are made before they get to 'processing' the problem – that is, a good half of the errors made were on account of difficulties with reading and comprehending the problem and transforming it into a mathematical context (NSW DEC, 2011).

Furthermore, the non-linear approach required in problem solving, combined with the necessity for a willingness to make mistakes, go back and to try again, often makes mathematical problem solving a challenging aspect of students' work.

Polya's 4 Step Problem Solving Process

Possibly the most well known version of the 'Problem Solving Cycle' is the one described by George Polya in his book '*How to Solve It'* (Polya, 1957).

Polya Taught four stages of problem solving. This four step process is very 'teachable' for most students from Year 3 onwards. Students can use Polya's method to consciously work on understanding the problem; to plan how they will 'attack' it mathematically; to decide upon a strategy and carry out the necessary mathematical operations; and finally, to check their solution for accuracy and relevance against the original problem. The following YouTube clip provides an overview of Polya's 'four-step problem solving process': https://www.youtube.com/watch?v=Cbw6-x8DPpQ

Students should be made aware that for many or most problems they will need to move 'back and forth' between different stages of the process before finally progressing to a solution. Remember, 'having a crack', making mistakes, going back and trying something different is all part of the process.

Furthermore, it's the 'Understanding' and 'Planning' aspects that students often become stuck on. Understanding what a question is asking and then planning which steps to take next require abstract thinking skills, as well as several – sometimes numerous – steps to follow. Students often and easily become 'lost in the detail' and so lots of practice with these stages of Polya's process is recommended.

Teaching Problem Solving as an Explicit Process

One practical suggestion might be to use a poster version of this on the classroom wall and use it to explicitly teach problem-solving as a process for students. Teachers could do this by working through the process using several examples, then putting students into groups to work through several more examples. In these stages, model for students the process of explicitly addressing each of these stages, reminding them that the focus of the task is on learning the process (not on 'answering the question in the quickest time possible').

Finally, set several examples for students to work through this process individually.

(ii) Types of Problem Solving

The diagram below (based from Foong Pui Yee, 2004) is one attempt to classify types of mathematical problem solving. What is apparent in this taxonomy of problem solving is the distinction made between 'Closed' and 'Open-Ended' problems.

Let's look at both in turn now.

Closed problems are "well-structured" in terms of clearly formulated tasks where the one correct answer can always be determined in some fixed way from the necessary data given in the problem situation.

Closed problems would include content-specific routine multiple-step problems as well as non-routine heuristic-based problems. ('Heuristic' implies that in the course of attempting or completing tasks, students learn concepts for themselves rather than having them explicitly explained or taught).

'Content-specific routine multiple-step problems' include those often used in textbooks, in which students are asked to solve a linear but multi-step problem within a specified content area.

Example

"Mara made a cake and cut it into 8 equal pieces. Her brother ate 1 piece. Mara took the remaining cake to a party, where her friends ate 5 pieces. She took home what was left. What fraction of the whole cake did Mara bring home?"

Non-routine closed problems emphasise the use of *heuristic* strategies (like *guess and check, work systematically, try simpler cases, tabulate data, look for a pattern, draw a diagram,* etc.) to solve an unfamiliar problem that is usually not content-specific to only one topic.

Non-routine closed problems are often useful for small group challenges or for work in pairs. This is because there are usually several ways in which to 'attack' these problems and it is good for students to recognise that there is a multiplicity of methods. Solutions to such non-routine problems, however, are still fixed (ie, there is a single mathematically correct answer to each part of the problem).

Example:

"I went shopping yesterday and my son Ben had to choose some cereal for breakfast. He wanted some Weet-Bix.

He said that his sister, Sarah-Rose, likes to eat two Weet-Bix a day and he eats three a day. No one else in our family eats Weet-Bix.

Now, Weet-Bix comes in boxes of 12, 24, 36, 48 and 72.

I wondered how many days the different boxes would last for and when I would have to buy more.

1) Write down how many days each box will last, feeding both kids;

2) Write down how many boxes will feed both kids each day over the period of exactly 4 weeks;

3) Write down how many Weet-Bix both kids would be able to eat each day if they were told that a large pack (72) was to last them for exactly 5 weeks and no longer. Would there be any left over?"

Open-ended Problems

By definition, open-ended questions and problems are usually missing data or assumptions, or at least leave some of these undefined. Furthermore, there is usually no fixed procedure or process that guarantees a correct solution. However, open ended mathematical problems do have an overall goal or required outcome and so do have 'incorrect' or inefficient solutions.

Many 'real-world problems' fall into this category, e.g. *'Design a better time-table for our school' or How much water and money can a school save during the "Save Water" campaign?*

'Open ended problems' are therefore mathematics problems that have a clear goal yet:

(a) leave one or more variables or assumptions open, undefined or to be determined by the problem-solver;

(b) have more than one solution (although some may be more efficient than others); and

(c) have more than one way of arriving at the overall goal.

Example:

"The toy shop stocks tricycles and go-carts.

The tricycles have 3 wheels.

The go-carts have 4 wheels.

Uh-oh. Someone pulled all the wheels off the go-carts and tricycles. There are 25 wheels altogether.

How many tricycles there? How many go-carts? Find different ways to do it." (DEE UK, 2000, p.38).

'Rich Tasks' and Mathematical Investigations

Variations on open ended problem solving tasks are mathematical investigations or so-called 'rich tasks' (sometimes referred to as 'authentic tasks'). These explore and extend a piece of pure mathematics for its own sake, present real-world problems that require mathematical modelling or integrate applied mathematical problems with other areas of curriculum or academic endeavour.

Such open-ended projects usually require students to demonstrate their ability in the form of a detailed report on how they carry out an extended piece of independent work in mathematics showing their creative application of mathematical knowledge and skills.

Rich tasks and investigations are a great source of alternative assessment in mathematics as they demonstrate not only whether students understand specific concepts, but also the extent to which they can apply those concepts to 'real world' type situations and to use them to solve unfamiliar problems. They allow teachers to provide solid assessment of the 'Working Mathematically' outcomes in the NSW Mathematics Syllabus (also known as the 'mathematical proficiencies' in the Australian Curriculum (Mathematics)).

Section 2: Why teach open-ended problem solving?

Let's now consider the benefits our students can gain from being exposed to open-ended problems in mathematics, and how these tasks can also help us in our mathematics teaching and assessment effort.

(i) Traditional approaches to mathematical questioning

David Tout, writing for ACER's 'Teacher Magazine', argues that *"In the typical mathematics classroom, especially in the middle years of schooling, we tend to use one model to connect maths with the real world; we start by teaching the maths content and skills, we then get students to practice and do some maths, and then we next might apply some of those skills into a real world context by using learning activities such as word problems."* [Tout, D. (2014)].

Note that there is nothing ‘wrong’ with this approach; indeed, it is necessary for developing understanding and fluency. It is, however, insufficient. What is doesn’t do is ‘stretch’ our students’ mathematical thinking beyond fluency with concepts into more creative problem solving and reasoning exercises.

In the real world, maths problems are not formulaic. They require us to extract the maths from ‘messy’ real-life situations and they often demand that we access more than one concept, operation, process or pathway to arrive at a solution.

By relying solely on a ‘teach-question-answer-correct’ approach to maths teaching, we inadvertently train our students to think that the ‘equals’ sign is mathematical code for ‘write the only correct answer’. This is a very binary way of thinking about mathematics (ie, ‘right’ or ‘wrong’). It hinders students from seeing that there are often several or more ways in which to solve real-world maths problems, and often more than one correct answer. It discourages students from ‘having a crack’ at solving a problem for fear of ‘getting it wrong’, and reinforces a fixed-mindset belief that they have either got it or they don’t, and therefore further exploration is a waste of time. (For further discussion on this issue in developing students’ thinking in mathematics, refer to Max Stephens’ work on Relational Thinking – Stephens 2012).

(ii) Why Use Open-Ended Problem Solving?

Traditional ‘drill and answer’ questions, and closed worded problems such as those typically used in textbooks and on online mathematics websites, don’t necessarily focus students’ learning on higher order thinking. They develop the proficiencies of understanding and fluency, and on communicating a correct answer using mathematical language and symbols.

In general, the more open-ended types of problems *do* have greater potential for stimulating higher order mathematical thinking. This is partly because they usually involve a search for patterns and relationships between elements in the problem. It is also because students must ‘play around with’ different variables in order to generate different solution pathways, trial and error and explore a range of methods, and use reasoning to apply learned concepts in a range of unfamiliar situations.

(iii) Benefits of Problem Solving for Students

In an article entitled “Why Teach Problem Solving?’, the New Zealand Ministry of Education’s ‘NZMaths’ website provides a detailed and compelling explanation of the benefits of problem solving in mathematics (http://nzmaths.co.nz/why-teach-problem-solving). They argue that problem solving in general, and open-ended problem solving in particular:

- produce more positive attitudes towards mathematics;
- teach thinking, flexibility and creativity;
- teach general problem solving skills;
- encourage cooperative skills; and
- is an interesting and enjoyable way to learn mathematics. (NZMaths, 2010).

Open-ended problem solving, including rich tasks and maths investigations, usually provide real world (if not simulated real-world) context for mathematical concepts. Context in maths for children and young adults is especially critical when seeking to engage them in our lessons.

Many children (and adults) immediately think only of slogging over a lead pencil and graph paper in a dreary classroom on a wet Monday when prompted by the word ‘mathematics’. Linking maths to practical situations, other areas of the curriculum or subjects of personal interest to students can go a long way toward improving student interest and engagement levels in mathematics (Garrett 2016).

The importance of teaching mathematical problem solving flows from its utility in the real world. It could be strenuously argued that without the application of mathematics to the resolution of real world problems, the discipline itself carries little significance.

Lastly, we need to consider the ‘big picture’ for our students. In an address concerning the importance of the introduction of the ‘Common Core’ for mathematics education in the United States, the respected and pre-eminent mathematics education expert and author Dr Jo Boaler of Stanford University made the following observation:

“The world is changing. We no longer need students to just be fast calculators… in fact, technology now does that for us. We need students who can think and develop mathematical models and reason and problem solve…” (Boaler, 2014)

Both now and in the future new and emerging fields of technology and science require the creative and sophisticated application of mathematical logical reasoning, problem solving and predictive modelling. We dare not let our students down by compromising on equipping them with the essential problem solving abilities and logical reasoning capacities that will help them to successfully meet these challenges.

(iv) Benefits for Teachers

There are a number of benefits of integrating and using open-ended problem solving in your mathematics classroom for you as a practitioner.

- ***A Straightforward Approach to Differentiated Learning.*** Firstly, a major advantage is that because there are multiple solutions, open-ended problems and investigations cater for a differentiated range of mathematical abilities and stages of development in children.

  The more able and experienced the child, the more sophisticated the investigation can become. However, being `successful' in open tasks can range from finding one possible solution in the form of a physical model and using concrete materials, to a systematic presentation and explanation of every possible solution (Way, 2005).

  Differentiation in the classroom, especially in groups or classes with a wide range of ability levels, is one of the most frequently cited concerns that teachers express about managing their professional practise. Open-ended tasks and investigations, by their nature, allow for a variety of ‘entry and exit points’ to the task and so one task will often provide appropriate challenge for a broader range of abilities than any set of closed activities or exercises (see McClure, 2011).

  The use of open tasks is in contrast to a more familiar procedure for differentiating mathematics instruction; that is, to break up a closed task that may be too difficult for some students and ask them to think about a few little bits of the task at a time; or, to simplify a closed task by using smaller numbers or by carefully scaffolding the way the question is presented. These approaches, while used for all of the right reasons, reinforce the idea that some students are not capable of independent mathematical thinking and denies some students opportunities to develop that capacity (LNS Ontario, 2008).

  In essence, open-ended tasks make differentiated learning much easier to plan and deliver because the use of open tasks is inclusive of all students’ mathematical thinking and is relative to students’ varied ‘zones of proximal development’.

Example:

Consider the following Grade 6 task:

"Choose one of these measurements:

- *1000 days,*
- *10 000 hours, or*
- *1 million seconds*

About how old is someone in your family using the measurement you unit chose?"

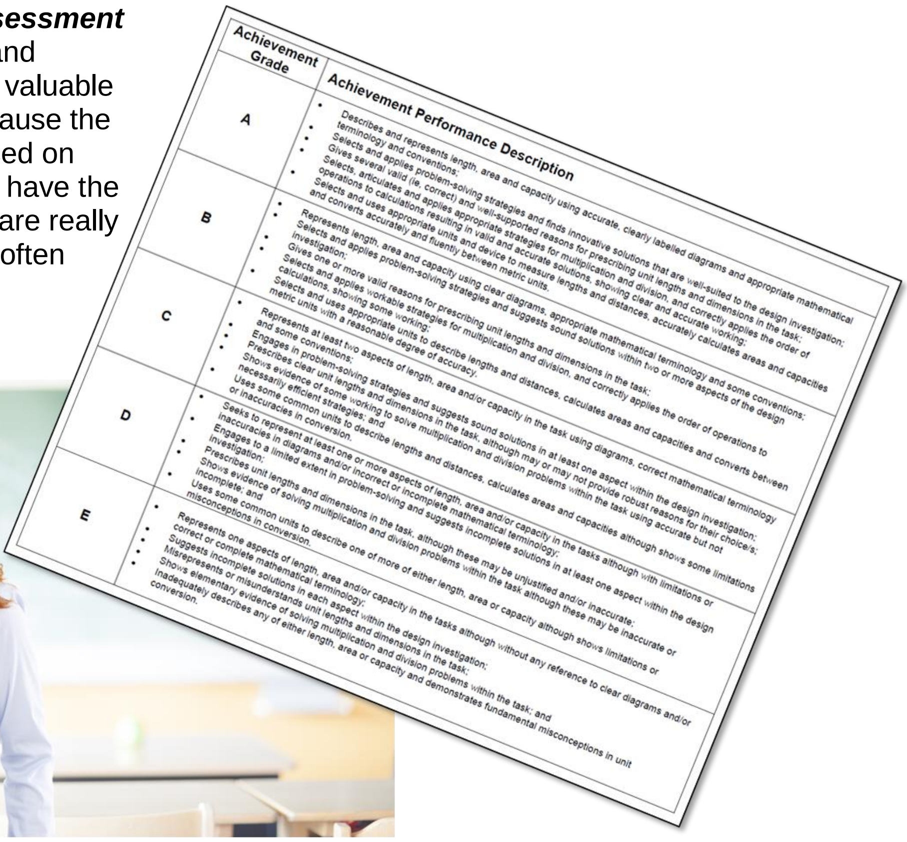
In this example, there are many ways to approach the task, and there is no single correct solution. Students can provide a solution relevant to their personal mathematics knowledge and experience and fully participate in a classroom discussion.

Similarly, rich tasks such as *'Number Walks'* or *'Park Ranger Problem Solving'* (http://calculate.org.au/category/curriclinks/) allow all students in a group to access the same domain of mathematics (ie, 'Counting On' or 'Addition and Subtraction' respectively), but individuals will access and work on the task at challenge levels appropriate to their abilities.

According to the Literacy and Numeracy Secretariat of Ontario in Canada (2008):

"The ultimate goal of differentiation is to meet the needs of the all students in a classroom during all parts of the problem-solving lesson. This becomes more manageable if the teacher can create a single task that allows not only different students to approach it using different processes or strategies, but also different students at different stages of mathematical development to benefit and grow mathematically."

- ***Provision for Alternative Assessment in Mathematics.*** Open tasks and investigations provide rich and valuable assessment opportunities, because the artificial limitations usually placed on children are removed and they have the opportunity to show what they are really capable of. Hidden talents are often revealed.



Achievement Grade	Achievement Performance Description
A	• Describes and represents length, area and capacity using accurate, clearly labelled diagrams and appropriate mathematical terminology and conventions; • Selects and applies problem-solving strategies and finds innovative solutions that are well-suited to the design investigation; • Gives several valid (ie, correct) and well-supported reasons for prescribing unit lengths and dimensions in the task; • Selects, articulates and applies appropriate strategies for multiplication and division, and correctly applies the order of operations to calculations resulting in valid and accurate solutions, showing clear and accurate working; • Selects and uses appropriate units and device to measure lengths and distances, accurately calculates areas and capacities and converts accurately and fluently between metric units.
B	• Represents length, area and capacity using clear diagrams, appropriate mathematical terminology and some conventions; • Selects and applies problem-solving strategies and suggests sound solutions within two or more aspects of the design investigation; • Gives one or more valid reasons for prescribing unit lengths and dimensions in the task; • Selects and applies workable strategies for multiplication and division, and correctly applies the order of operations to calculations, showing some working; • Selects and uses appropriate units to describe lengths and distances, calculates areas and capacities and converts between metric units with a reasonable degree of accuracy.
C	• Represents at least two aspects of length, area and/or capacity in the task using diagrams, correct mathematical terminology and some conventions; • Engages in problem-solving strategies and suggests sound solutions in at least one aspect within the design investigation; • Prescribes clear unit lengths and dimensions in the task, although may or may not provide robust reasons for their choice/s; • Shows evidence of some working to solve multiplication and division problems within the task using accurate but not necessarily efficient strategies; and • Uses some common units to describe lengths and distances, calculates areas and capacities although shows some limitations or inaccuracies in conversion.
D	• Seeks to represent at least one or more aspects of length, area and/or capacity in the tasks although with limitations or inaccuracies in diagrams and/or incorrect or incomplete mathematical terminology; • Engages to a limited extent in problem-solving and suggests incomplete solutions in at least one aspect within the design investigation; • Prescribes unit lengths and dimensions in the task, although these may be unjustified and/or inaccurate; • Shows evidence of solving multiplication and division problems within the task although these may be inaccurate or incomplete; and • Uses some common units to describe one of more of either length, area or capacity although shows limitations or misconceptions in conversion.
E	• Represents one aspects of length, area and/or capacity in the tasks although without any reference to clear diagrams and/or correct or complete mathematical terminology; • Suggests incomplete solutions in each aspect within the design investigation; • Misrepresents or misunderstands unit lengths and dimensions in the task; • Shows elementary evidence of solving multiplication and division problems within the task; and • Inadequately describes any of either length, area or capacity and demonstrates fundamental misconceptions in unit conversion.

One possibility is that open-ended problem solving can be graded in alignment with A-E marking rubrics using the stage-specific 'Working Mathematically' outcomes that apply to the relevant grade or stage (included along with the content-specific outcome statements that apply to the task topic). Once a marking rubric has been developed for an open-ended problem or rich task investigation, the process of marking and assessment for the task becomes straightforward and efficient. Teachers are then provided with valid and substantiated evidence for reporting on students' performance against Working Mathematically outcomes.

Multiple solutions and provide a rich source of material for mathematical discussion, which adds depth to teacher's insight regarding what students know (and on any misconceptions they might have), further adding to their bank of assessment information.

- ***Student Engagement in mathematics and Classroom Management.*** To the extent that open-ended problems and investigative tasks can increase student engagement, these aspects of mathematics teaching and learning are inherently more interesting to teach and more enjoyable to manage. Engaged and on-task children are happy and make managing a class a much more pleasant experience.

  Of course, an open-ended problem solving approach in your maths lessons is not and will never be a 'magic bullet' for solving those difficult classroom management cases you may be dealing with – but it will at least increase your chances of engaging students in mathematics who might otherwise be easily distracted during more routine 'drill and answer' exercises from a textbook or whiteboard (see also Jo Boaler's book *'The Elephant in the Classroom'*, 2010).

  NZMaths has this to say about open-ended problem solving and student engagement:

  "Problem solving seems to employ problems that are implicitly interesting to children. This is partly because problem solving does not involve a sequence of very similar questions that are designed to practice the same skill. The novelty of the problems seems to add to their interest.

  Many teachers personalise word problems to include characters that the children in the class know. This also makes them more interesting and relevant to the children.

  Then again the questions can be very interesting in themselves. This is partly because they involve some detective work, which most people enjoy. It's also partly because we all enjoy getting the answer after having struggled with a problem. And it's partly because children enjoy having "ownership" of the problem. The ownership issue is an important one. By working on a problem, children become involved with it and can get quite deeply involved with the mathematics that is both required to solve it, and that may be required to solve it." (NZMaths 2010)

  We'll now consider how we can incorporate open-ended problem solving into mathematics lessons, as well as sources of open-ended problems and ways in which to develop our own open-ended problems and activities.

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