Arithmetic to Algebra
through the Australian Curriculum

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What is Arithmetic?

Numbers

Addition

Multiplication

Division

Fractions

Subtraction

Decimals

Place Value
What is Algebra?

Letters

Variables

Equations

Formulas

Solving Unknowns

Substitution

Transposition
When does algebra start in the Australian Curriculum?
<table>
<thead>
<tr>
<th>Year</th>
<th>Summary</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Foundation</strong></td>
<td>students make connections between number names, numerals and quantities up to 10. Students count to and from 20 and order small collections.</td>
</tr>
<tr>
<td><strong>Year 1</strong></td>
<td>students describe number sequences resulting from skip counting by 2s, 5s and 10s. They identify representations of one half. Students count to and from 100 and locate numbers on a number line. They carry out simple additions and subtractions using counting strategies. They partition numbers using place value. They continue simple patterns involving numbers and objects.</td>
</tr>
<tr>
<td><strong>Year 2</strong></td>
<td>students recognise increasing and decreasing number sequences involving 2s, 3s and 5s. They represent multiplication and division by grouping into sets. Students identify the missing element in a number sequence. Students count to and from 1000. They perform simple addition and subtraction calculations using a range of strategies. They divide collections and shapes into halves, quarters and eighths. Students order shapes and objects using informal units.</td>
</tr>
<tr>
<td><strong>Year 3</strong></td>
<td>students recognise the connection between addition and subtraction and solve problems using efficient strategies for multiplication. They model and represent unit fractions. Students count to and from 10 000. They classify numbers as either odd or even. They recall addition and multiplication facts for single digit numbers. They continue number patterns involving addition and subtraction.</td>
</tr>
<tr>
<td><strong>Year 4</strong></td>
<td>students choose appropriate strategies for calculations involving multiplication and division. They recognise common equivalent fractions in familiar contexts and make connections between fraction and decimal notations up to two decimal places. They identify unknown quantities in number sentences. They describe number patterns resulting from multiplication. Students use the properties of odd and even numbers. They recall multiplication facts to 10 x 10 and related division facts. Students locate familiar fractions on a number line. They continue number sequences involving multiples of single digit numbers.</td>
</tr>
<tr>
<td><strong>Year 5</strong></td>
<td>students solve simple problems involving the four operations using a range of strategies. They check the reasonableness of answers using estimation and rounding. Students identify and describe factors and multiples. Students order decimals and unit fractions and locate them on number lines. They add and subtract fractions with the same denominator. Students continue patterns by adding and subtracting fractions and decimals. They find unknown quantities in number sentences.</td>
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<tr>
<td><strong>Year 6</strong></td>
<td>students recognise the properties of prime, composite, square and triangular numbers. They describe the use of integers in everyday contexts. They solve problems involving all four operations with whole numbers. Students connect fractions, decimals and percentages as different representations of the same number. They solve problems involving the addition and subtraction of related fractions. Students make connections between the powers of 10 and the multiplication and division of decimals. They describe rules used in sequences involving whole numbers, fractions and decimals. Students connect decimal representations to the metric system and choose appropriate units of measurement to perform a calculation. Students locate fractions and integers on a number line. They calculate a simple fraction of a quantity. They add, subtract and multiply decimals and divide decimals where the result is rational. Students calculate common percentage discounts on sale items. They write correct number sentences using brackets and order of operations.</td>
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<td><strong>Year 7</strong></td>
<td>students solve problems involving the comparison, addition and subtraction of integers. They make the connections between whole numbers and index notation and the relationship between perfect squares and square roots. They solve problems involving percentages and all four operations with fractions and decimals. <strong>Students represent numbers using variables. They connect the laws and properties for numbers to algebra.</strong> They interpret simple linear representations and model authentic information. Students use fractions, decimals and percentages, and their equivalences. They express one quantity as a fraction or percentage of another. Students solve simple linear equations and evaluate algebraic expressions after numerical substitution.</td>
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<tr>
<td><strong>Year 8</strong></td>
<td>students solve everyday problems involving rates, ratios and percentages. They recognise index laws and apply them to whole numbers. They describe rational and irrational numbers. They <strong>make connections between expanding and factorising algebraic expressions.</strong> Students use efficient mental and written strategies to carry out the four operations with integers. They <strong>simplify a variety of algebraic expressions.</strong> They <strong>solve linear equations</strong> and graph linear relationships on the Cartesian plane.</td>
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Is it this simple?

What about:

Order of operations

Recognising patterns

Describing rules

Working with unknowns
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Important arithmetic ideas for algebra: Laying the foundations in the upper primary years

Very relevant to all of primary and junior secondary as well
Algebra is a language
- a way of saying and communicating.

Algebra is more succinct than English
- making it easier to manipulate
- but also more open to incomprehension
Algebra is a powerful means of communicating abstract and complex ideas.

It has its own rules which must be learnt and practised.

It is an ideal way to see and express general statements.
Consider the consecutive numbers
10, 11, 12

Multiply the 1st and 3rd numbers
10 x 12 =

Now, square the middle number
11 x 11 =

Where would we place this in the curriculum?
Year 4
What is the difference between the answers?

\[ 10 \times 12 = 120 \]

\[ 11 \times 11 = 121 \]

Difference is 1

Extending thinking: Does this always work for consecutive whole numbers?
We can use algebra...

Let $x, x + 1, x + 2$ represent three consecutive whole numbers
I want to show that the difference between the square of the middle number and the product of the first and third, is one.

\[
\text{I want } (x + 1)^2 - x (x + 2) = 1
\]

For our number example:
\[
(10+1)^2 - (10) (10 + 2) = 1
\]
It does not take much manipulation to see that the result *will* always be true:
\[(x + 1)^2 - x (x + 2) = 1\]

Why stop here? Why not test consecutive even numbers, or consecutive odd numbers...
Algebra

Has had some bad press
- Algebra is hard
- When will I ever use it?
- Where does algebra have applications in my life?
Perhaps one reason for the *attitude* people have to algebra is that it has been taught without the links to arithmetic being made explicit.
Students need to develop an overall framework to help them make sense of how the various parts fit together and the purpose of them.
Why teach it?

As well as simply being part of a well rounded mathematical education, Algebra supports

- problem solving
- logical deduction
- abstraction
- seeing and expressing generalisation
Addition
Any order principle of addition

\[ 13 + 25 + 45 + 27 \]
• Why is it important to think about arithmetic in this way?

• Does it apply only to addition?
Any order principle of addition

3x + 4y + 7x + 11y
Any order principle of multiplication

$25 \times 7 \times 4 \times 3$
Any order principle of multiplication

\[ 3xy^2 \times 4x2y^3 \]
Example

Using the number line

The order in which we perform addition does not matter. For example, $6 + 4 = 4 + 6$. This can be shown on the number line.

$6 + 4 = 10$

$4 + 6 = 10$

This property is called the **commutative law for addition**.
We can produce a long list of arithmetic statements such as

\[
4 + 3 = 3 + 4
\]
\[
2 + 6 = 6 + 2
\]
\[
8 + 3 = 3 + 8
\]

Each is an example of the commutative property of addition.
One algebraic statement defines the commutative law

\[ a + b = b + a \]

where \( a \) and \( b \) are whole numbers.
Importance of building blocks and sequencing

- The teaching of number and algebra are inextricably linked.

We cannot expect an improvement in student’s learning of algebra until we succeed in building their understanding of arithmetic, i.e. knowledge of number and number operations, and mental computation techniques.
Do this calculation in your head

$3456789 \times 42 + 3456789 \times 58$

And the answer is:
‘A strong grounding in high school mathematics through Algebra correlates powerfully with access to college, graduation from college, and earning in the top quartile of income from employment. The value of such preparation promises to be even greater in the future’.
Claims based on Piaget’s highly influential theory, and related theories of “developmental appropriateness” that children of particular ages cannot learn certain content because they are “too young,” “not in the appropriate stage,” or “not ready” have consistently been shown to be wrong. Nor are claims justified that children cannot learn particular ideas because their brains are insufficiently developed, even if they possess the prerequisite knowledge for learning the ideas’.
The teacher wrote an open number sentence: $7 + 6 = \Box + 5$, and asked children to find the missing number and to say how they found it.

Here are four different responses:
Teacher: Luke, what number did you put in the box?

Luke: Thirteen

Teacher: How did you decide?

Luke: 7 and 6 are 13

Teacher: What about the 5?

Luke: It doesn’t matter. The answer to 7 + 6 is 13

Teacher: What is the 5 doing then?

Luke: It’s just there.
Teacher: Cameron, what number did you put in the box?

Cameron: Eighteen

Teacher: How did you decide?

Cameron: 7 and 6 are 13 and 5 more is 18

Teacher: Does 7 plus 6 equal to 18 plus 5?

Cameron: 7 + 6 is 13 and 5 more is 18
Teacher: Fiona, what number did you put in the box?

Fiona: Eight

Teacher: How did you decide?

Fiona: 7 and 6 gives 13 and I then thought what number goes with 5 to give 13.

7 + 6 is 13 and 5 + 8 is 13
7 + 6 = $\Box$ + 5

**Teacher:** Chris, what number did you put in the box?

**Chris:** Eight

**Teacher:** How did you decide?

**Chris:** (Points to the numbers)

$$7 + 6 = \Box + 5$$

5 is one less than 6, so you need a number that is one more than 7 to go in the $\Box$ so it all balances.
Each child interpreted the number sentence and the equal sign in their own way.

Luke and Cameron made reasonable attempts to deal with an unfamiliar problem.

They appear to think that the equal sign is always followed by an answer.

The equal sign is being used in a quite different way in these expressions.
Fiona and Chris see the equal sign as expressing an equivalence between the numbers represented on both sides.

They can accept that the equal sign is not always followed by an answer.

They are both comfortable in having number sentences with different forms.

Children need to meet number sentences with different forms and with different operations.
Fiona added both numbers on the left side, and then looked for a number to place in the box that would give the same total. Perhaps she thought:

\[ 7 + 6 = \square + 5 \]
\[ 7 + 6 = 13, \text{ so} \]
\[ 13 = \square + 5, \text{ so} \]
what do I have to add to 5 to get 13. Yes 8.

She sees that the results of the calculations on both sides have to be the same.
Chris’ method is subtly different.

5 is one less than 6, so you need a number that is one more than 7 to go in the □ so it all balances.

She looks at the relation between the two addition expressions on either side of the equal sign, not just at the answers of the two calculations.

What would you ask to see if Chris is a confident user of this kind of thinking?

It is possible that Fiona can think this way too – we don’t know for sure – but it is unlikely that Luke or Cameron can think in this way without further teaching.
The ability to identify and use relations within number sentences such as:

\[ 7 + 6 = \Box + 5, \text{ or } \]
\[ 27 + \Box = 30 + 5 \]

is very important for students in the primary years:

- to think about the **structure** of arithmetic relations, and
- to use that knowledge as a **bridge to algebra**
Loretta has written the following number sentence

$$34 + 29 = 33 + 30$$

She did not have to add up the numbers to know this. Why?
One Year 6 student said, “Loretta just knows that they both add up to 63”.

A Year 5 student said: “Loretta can do this because she did it in her head”.

Neither student could explain why Loretta did not need to add the numbers in order to know that she was correct without using an explanation based on computation.
One Year 5 student drew the following:

\[ 34 + 29 = 33 + 30 \]

Referring to the 29, the student wrote, “It increases by 1 to give 30, so 34 has to decrease by 1 to give 33”.

A second Year 5 student inserted an arrow:

\[ 34 + 29 = 33 + 30 \]

No number was attached to the arrow, but the student wrote: “If one unit moves from the 30, the other number becomes 34”
Introducing young children to relational thinking is not an easy task when teachers’ vision has for so long been restricted to thinking of arithmetic as calculation.

In the primary school, this means attending to the structure of arithmetic operations.

Without these experiences, many students fail to understand these structures necessary for a successful transition to algebra.
Teachers could introduce number sentences involving subtraction (difference) such as:

- $41 \square 15 = 43 \square \square$
- $104 \square 45 = \square \square 46$

Use “difference” rather than “subtraction”

Represent this difference on a number line
How would you calculate?

3000

−1563
Can relational thinking help us to find alternatives?

\[ 3000 \]

\[ -1563 \]

Some students find algorithms for subtraction (e.g. method of \textit{decomposition}) difficult and time consuming.
Given 3000 – 1563

Increase both numbers by 37 gives

3037 – 1600

This can be calculated more easily!! 1437

(Why did we choose 37? Because it makes the second number 1600 and easier to subtract)
We can also change the first number:

Given 3000 – 1563

Decrease both numbers by 1: 2999 – 1562

2999

– 1562

1437

This can be calculated more easily!! 1437
• Is a powerful way of drawing attention to some fundamental **structures** of arithmetic

• Two key ideas are:
  – **equivalence** of expressions, and
  – **compensation**, including knowing the direction in which compensation takes place

• These ideas also provide a foundation for **algebraic thinking**
### Keeping the PRODUCT the same

#### Multiplication

<table>
<thead>
<tr>
<th>Original</th>
<th>New</th>
<th>Product</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1st number</strong></td>
<td><strong>2nd number</strong></td>
<td><strong>1st number</strong></td>
</tr>
<tr>
<td>16</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>25</td>
<td>12</td>
<td>100</td>
</tr>
<tr>
<td>0.4</td>
<td>600</td>
<td>4</td>
</tr>
<tr>
<td>24</td>
<td>2/3</td>
<td>8</td>
</tr>
</tbody>
</table>
Keeping the QUOTIENT the same

<table>
<thead>
<tr>
<th>Original</th>
<th>New</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st number</td>
<td>2nd number</td>
</tr>
<tr>
<td>16</td>
<td>5</td>
</tr>
<tr>
<td>270</td>
<td>6</td>
</tr>
<tr>
<td>14000</td>
<td>7000</td>
</tr>
<tr>
<td>0.8</td>
<td>0.2</td>
</tr>
<tr>
<td>0.18</td>
<td>0.02</td>
</tr>
<tr>
<td>7</td>
<td>1/3</td>
</tr>
<tr>
<td>7</td>
<td>2/3</td>
</tr>
</tbody>
</table>
### Comparing Addition and Subtraction (inverse operations)

<table>
<thead>
<tr>
<th>Changes to 1&lt;sup&gt;st&lt;/sup&gt; number</th>
<th>Changes to 2&lt;sup&gt;nd&lt;/sup&gt; number</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>+5</td>
<td>-5</td>
<td>same</td>
</tr>
<tr>
<td>-3.5</td>
<td>+3.5</td>
<td>same</td>
</tr>
<tr>
<td>-1/4</td>
<td>+1/4</td>
<td>same</td>
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</table>

#### Addition

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<tr>
<th>Changes to 1&lt;sup&gt;st&lt;/sup&gt; number</th>
<th>Changes to 2&lt;sup&gt;nd&lt;/sup&gt; number</th>
<th>Difference</th>
</tr>
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<tr>
<td>+5</td>
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<td>same</td>
</tr>
<tr>
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</tr>
<tr>
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# Comparing Multiplication and Division (inverse operations)

## Multiplication

<table>
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<tr>
<th>Changes to 1st number</th>
<th>Changes to 2nd number</th>
<th>Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>x 5</td>
<td>÷ 5</td>
<td>same</td>
</tr>
<tr>
<td>÷ 25</td>
<td>x 25</td>
<td>same</td>
</tr>
<tr>
<td>÷ 10</td>
<td>x 10</td>
<td>same</td>
</tr>
</tbody>
</table>

## Division

<table>
<thead>
<tr>
<th>Changes to 1st number</th>
<th>Changes to 2nd number</th>
<th>Quotient</th>
</tr>
</thead>
<tbody>
<tr>
<td>x 5</td>
<td>x 5</td>
<td>same</td>
</tr>
<tr>
<td>÷ 25</td>
<td>÷ 25</td>
<td>same</td>
</tr>
<tr>
<td>x 10</td>
<td>x 10</td>
<td>same</td>
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## Summary of 4 operations

<table>
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<tr>
<th>Operation in number sentence</th>
<th>Keep this the same</th>
<th>Direction of adjustments* to the original numbers</th>
<th>Operations involved in these adjustments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition</td>
<td>Sum</td>
<td>Opposite directions</td>
<td>Additive (+/-)</td>
</tr>
<tr>
<td>Subtraction</td>
<td>Difference</td>
<td>Same direction</td>
<td>Additive (+/-)</td>
</tr>
<tr>
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<td>Product</td>
<td>Opposite directions</td>
<td>Multiplicative (x/÷)</td>
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<tr>
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<td>Quotient</td>
<td>Same direction</td>
<td>Multiplicative (x/÷)</td>
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* Note that the numbers involved in these adjustments can be any type of number (eg whole numbers, fractions, decimals)
We can keep the sum or the difference the same by making particular adjustments to each number using the operations of addition and subtraction.

- To keep the sum the same, one number is increased by a certain amount and the other number is decreased by that same amount.

- To keep the difference the same, both numbers are increased (or decreased) by the same amount.

In each case, the amount of increase or decrease can be any type of number (whole, fraction, decimal) and the type of increase or decrease is “additive” (which means involves only addition or subtraction).
Summarising what we found \( x \div \)

We can keep the **product** or the **quotient** the same by making particular adjustments to each number using the operations of **multiplication** and **division**.

- To keep the **product** the same, one number is increased by a certain amount and the other number is decreased by that same amount.
- To keep the **quotient** the same, both numbers are increased (or decreased) by the same amount.

In each case, the amount of increase or decrease can be **any type of number** (whole, fraction, decimal)

In each case, the type of increase or decrease is “**multiplicative**” (which means involves only multiplication or division)
Fact Families and contexts for operations

- Students should be able to obtain related facts from a single number fact:
  - E.g., from $6 \times 5 = 30$ we also know
    
    $5 \times 6 = 30$
    
    $30 \div 6 = 5$
    
    $30 \div 5 = 6$

- They should also have experience of relating a worded situation to a numerical expression
  
  - E.g., the number sentence $6 \times 5 = 30$ could arise from “there are 6 tables, and each table has 5 people sitting at it, so there are 30 people all together”
Inverse Operations

- Understand the relationship between addition/subtraction and multiplication/division and even square/square root

- Equivalence of

  \[ 12 + 8 = 20 \]
  \[ \leftrightarrow \]
  \[ 20 - 8 = 12 \]

  \[ 6 \times 4 = 24 \]
  \[ \leftrightarrow \]
  \[ 24 \div 4 = 6 \]

  \[ 7^2 = 49 \]
  \[ \leftrightarrow \]
  \[ \sqrt{49} = 7 \]
Role of the Equals Sign

• As teachers you need to set the example.
• Only write the equals sign between quantities that are equal.
• Don’t use the equals sign for “run on calculations”.
  • E.g., if solving “a boy has five marbles and then a friend gives him ten more, and then he loses two” then do NOT write
    \[ 5 + 10 = 15 - 2 = 13 \]
Number Properties

• Need to know the multiples

• Need to know how to factorise numbers (and then rearrange the factors)

  e.g., 24 = 2 \times 2 \times 2 \times 3
  = 8 \times 3 = 6 \times 4 = 12 \times 2

• Useful to know the square numbers:
  1, 4, 9, 16, 25, 36, ...
Students need to be familiar with

• arithmetic
• patterns in arithmetic
• relationships between numbers and operations
While arithmetic is still being consolidated, care should be taken not to overdo algebra.

For example,

20% of an amount is $6, what is 100%?
The tendency might be to use algebra.

\[
20\% \text{ of } x = 6 \\
\frac{x}{5} = 6 \\
x = 30
\]
I prefer the unitary method

10% is $3

100% is $30
Number is an abstract construct of the human mind. Like anything abstract it is difficult to grasp – literally! In order to use the concept of numbers effectively we need to internalise their properties, they need to become second nature to us.
Today’s number

360

Tools session idea:
*Today’s number is…handout-MCTP*
Today’s number

Use often to
- Reinforce number facts
- Engage students in thinking about strategies
- Allow students to pull apart and then reconstruct numbers

Use it to
- Settle and focus students
- Allow all students time to respond.
- Pre-test and post-test by counting number of facts in a minute with extra points for FAT facts.
- Insist on certain strategies. For example, students must include multiplying with fractions.
Number line

Use the number line to illustrate
• Order
• Addition
• Subtraction
• Multiplication
• Division

Step through introduction to number lines - do not assume students have used them before.
Number lines are used later to illustrate similar properties for fractions, decimals and integers.
Number line

When internalising the number line, give attention to

Position - particularly starting position

Orientation – positive or negative

Direction of travel – positive or negative

Mode of travel – what operations are being used
### Ideas - lesson events

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Ideas - lesson events

- Hatsumon (asking key questions)
- Kikan-shido (between the desks instruction)
- Neriage (polishing student ideas during discussion)
- Matome (summarizing the learning)
Ideas - kikan shido

Japanese for *Between the desks instruction*

Professor David Clarke
International Centre for Classroom Research (ICCR)
University of Melbourne

Observed in Australian, Chinese and Japanese classrooms to:
- monitor students’ current activities
- keep students on task - proximity
- check homework is complete
- question, prompt, and generally scaffold students’ activity
- explicit demonstration by teacher
- reminding of rules
- inspecting around
- giving voice to students’ ideas
- encouraging and motivating students
- gauge success of teacher presentation of ideas
Questions from the pre-test

Volunteers to demonstrate?
Multiplication - More than repeated addition
More than repeated addition
Multiplication- arrays

15
Multiplication - arrays

15
The factors of 15 are 3 and 5
1 and 15
Multiplication

Linking arrays and areas with the multiplication algorithm.
For example, $8 \times 17$
Multiplication

17

8
Multiplication

\[ 8 \times 10 + 8 \times 7 = 8 \times 17 \]

\[ = 80 + 56 \]

\[ = 136 \]
Multiplication

\[
\begin{array}{ccc}
5 & 1 & 7 \\
\times & 8 \\
\hline
1 & 3 & 6 \\
\end{array}
\]
Multiplication

Linking arrays and areas with the long multiplication algorithm.
For example, $27 \times 13$
Draw an array
Multiplication

Highlight the chunks

$10 \times 20$  $10 \times 7$

$3 \times 20$  $3 \times 7$
Multiplication

10 times 7 is 70
10 times 20 is 200
3 times 7 is 21
3 times 20 is 60 plus 20 is 80

Algorithm

\[
\begin{array}{c}
227 \\
\times \ 13 \\
81 \\
270 \\
\hline
351
\end{array}
\]
Area model and multiplication are fundamentally linked. It can be used in the introduction of algebra but must be based on a sound understanding of area and multiplication with numbers.
Algebra - Area

\[(a+b)^2 = a^2 + 2ab + b^2\]
More than repeated addition
Multiplication

Use a variety of multiplication models as well as the algorithm.
• Number line
• Arrays
• Area

Topping up skills idea:
Understanding and rapid recall of multiplication tables is important.
Multiplication

Distributive law

23 \times 6 = (20 + 3) \times 6

= 20 \times 6 + 3 \times 6
= 120 + 18
= 138

6 \times 27 = 6 (20 + 7)
= 6 \times 20 + 6 \times 7
= 120 + 42
= 162
Use place value to extend known number facts

If we know $5 \times 7$, then we also know $50 \times 7$

One of our numbers is ten times bigger than our number fact, so our answer will be ten times bigger. Ten times 35, that’s 350

$50 \times 70$

We say “five times seven is 35, each number is ten times bigger than our number fact so our answer will have to be $10 \times 10$ bigger. One hundred times 35 is 3500.
Mental strategies

Mentally multiplying by powers of 2

· 4
As $4 = 2 \times 2 (= 2^2)$ multiplying by 4 can be completed by multiplying by 2 twice. That is, you double twice.
For example
$34 \times 4 = 34 \times 2 \times 2 = 68 \times 2 = 136$

· 8
As $8 = 2 \times 2 \times 2 (= 2^3)$ multiplying by 8 can be completed by multiplying by 2 three times. That is, you double, then double once more and then double again.
For example
$13 \times 8 = 13 \times 2 \times 2 \times 2 = 26 \times 2 \times 2 = 52 \times 2 = 104$
Mental strategies

Mentally multiplying by 5

As $5 = 10 \div 2$, first multiply by 10 and then divide by 2.
$36 \times 5 = 36 \times 10 \div 2 = 360 \div 2 = 180$

Or divide by 2 and multiply by 10
$36 \div 2 \times 10 = 18 \times 10 = 180$
Mental strategies

Mentally grouping powers of 5’s and 2’s

\[25 \times 14 = 25 \times 2 \times 7 = 50 \times 7 = 350\]

\[75 \times 6 = 75 \times 2 \times 3 = 150 \times 3 = 450\]

\[125 \times 16 = 100 \times 16 + 25 \times 4 \times 4 = 1600 + 400 = 2000\]
Mental strategies

Mentally dividing by 5
To divide by 5, divide by 10 and then double

\[ 450 \div 5 = 45 \times 2 \]

\[ = 90 \]

Mentally dividing by 4
Halve and the halve again

\[ 428 \div 4 = 214 \div 2 \]

\[ = 107 \]
Mental strategies

Several divisions done mentally

In order to simplify computations it is sometimes easier to perform a chain of divisions. For example, instead of $18 \div 6$
consider $(18 \div 2) \div 3 = 9 \div 3 = 3$

The use of brackets is not necessary but at this stage makes the process clearer. It can be used to aid mental calculation.
Mental strategies

Using doubling and halving
Sometimes it is possible to multiply two numbers by doubling one number and halving the other. Keep doubling and halving until you get more manageable numbers.

\[34 \times 5 = 17 \times 10 = 170\]

\[28 \times 3 = 14 \times 6 = 7 \times 12 = 84\]
Mental strategies

Multiplying by 9 mentally
To multiply a number by 9, multiply by ten and take away one of the number.

\[
128 \times 9 = 128 \times (10 - 1) \\
= 128 \times 10 - 128 \\
= 1280 - 128 \\
= 1152
\]
Mental strategies

**Multiplying two digit numbers by 11 mentally**
To multiply a number by 11, multiply by ten and add one of the number.

\[
128 \times 11 = 128 \times 10 + 128 \\
= 1280 + 128 \\
= 1408
\]
Division

Models for division
• Arrays
• Number line

Other division ideas to consider
• Long division
  – more natural
  – explains process to a greater degree
  – Use as a demonstration of an idea rather than an assessable skill
• Short division
Prime numbers

Manhattan
On a unifix 100 board:
• place orange unifix cubes on all the multiples of 2
• place blue unifix cubes on all the multiples of 3
• place red unifix cubes on all the multiples of 4
And so on up to multiples of 10

Prime numbers have no cubes on them
Numbers with lots of factors are the tallest ‘buildings’

With thanks to Mark Richardson,
Williamstown Primary School, Victoria

The Licorice Factory handout - MCTP
Prime factorisation

Factor trees do not give all factors of a number, but can be used to find prime factors.

Prime numbers

*The Licorice Factory* handout - MCTP
Prime factorisation

Division by prime numbers gives prime factors.

\[
\begin{array}{c|c}
2 & 24 \\
2 & 12 \\
3 & 6 \\
3 & 3 \\
1 & 1 \\
\end{array}
\]
Prime factorisation

Consider 10! using prime factorisation.

\[ 10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \]
\[ = 2 \times 5 \times 3^2 \times 2^3 \times 7 \times 2 \times 3 \times 5 \times 2^2 \times 3 \times 2 \]
\[ = 7 \times 5^2 \times 3^4 \times 2^8 \]

Now we can ask questions such as how many zeroes there would be at the end of such a number.

Try 90! It doesn’t take as long as you think!
Lowest common multiple

Using prime factorisation.
For example, find the lowest common multiple of 24 and 18.

\[ 24 = 2^3 \times 3 \quad 18 = 2 \times 3^2 \]

For the LCM, take the factors with the highest power from each.

\[ \text{LCM} = 2^3 \times 3^2 \]

\[ = 8 \times 9 \]

\[ = 72 \]

When am I going to use this?
Needed for fractions and algebra
Using prime factorisation.

For example, find the highest common factor of 24 and 18.

\[ 24 = 2^3 \times 3 \]
\[ 18 = 2 \times 3^2 \]

For the HCF, take the factors with the lowest power from each.

\[ \text{HCF} = 2 \times 3 \]
\[ = 6 \]
Prime factorisation - alternative method

Korean method - (Christian bros Qld method)

To find the lcm and hcf of 24 and 36:

\[
\begin{array}{c|cc}
2 & 24 & 36 \\
\hline
2 & 12 & 18 \\
3 & 6 & 9 \\
\hline
& 2 & 3 \\
\end{array}
\]

HCF = \(2^2 \times 3\) = 12

LCM = \(2^2 \times 3 \times 2 \times 3\) = 72
Square root

Square roots by prime factorisation.

To find the square root of 1296, first find the prime factors.

\[
\sqrt{1296} = \sqrt{2^4 \times 3^4} \\
= \sqrt{2^2 \times 3^2} \times \sqrt{2^2 \times 3^2} \\
= 2^2 \times 3^2 \\
= 36
\]
Mental arithmetic - Multiplication and division

It is all about using and understanding the most efficient methods.

Handy to have good numbers for students to use these strategies on.
Fractions

• Defining fractions on number line

• A fraction is both
  – A point on the number line
  AND
  – The distance from 0, a length
Fractions

To represent thirds on a number line
• Draw a line segment
• Mark in 0, 1, 2 etc
• Divide the segment into three equal lengths
• Label each marker one third, two thirds etc
Fractions - equivalence

Two fractions are equivalent when they define the same point on the number line.

A definition is needed.
Fractions - area

Based on a unit square:
• Side length = 1
• Area = 1
Fractions - area

Shade parts of the unit square
• Denominator gives number of parts
• Numerator tells us “How many parts to take”
Fractions - multiplication

\[
\frac{2}{3} \times \frac{3}{4}
\]

\[
\frac{3}{4}
\]

\[
\frac{2}{3}
\]
Fractions - multiplication

\[
\begin{array}{ccc}
\frac{3}{4} & & \\
& & \\
& & \\
\end{array}
\]
Fractions - multiplication

\[ \frac{2}{3} \text{ of } \frac{3}{4} \]
Fractions - multiplication
Fractions - multiplication

\[
\frac{3}{4} \text{ of } \frac{2}{3}
\]
Fractions - multiplication

\[
\frac{5}{4} \times \frac{3}{2}
\]