‘How Long Is a Piece of String?’ *(Part 2)*

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This is the second part of our feature article on open-ended problem solving. It will focus on some practical ways we can develop and implement open-ended problem solving in our mathematics lessons.

1. Building Open-Ended Problem-Solving into Lesson Structure

There are a number of ways in which teachers can build open-ended problem solving into their lesson structure. Here are several suggestions:

a) ***Using open-ended problems and puzzles as lesson ‘warm-ups’.*** Setting single whole-class or small group open ended tasks as mathematical ‘warm-ups’ to start lessons can be a great way to get students thinking and working mathematically. Spend a strictly-timed 5 to 10 minutes at the start of the lesson having students work on an open-ended puzzle. Some students may work towards finding only one solution to the puzzle; others may find multiple solutions in the time set.

Each time you use this strategy as a warm-up, ask three students (and three students only) how they set about trying to solve the puzzle. This will start your maths session with a brief mathematical discussion and ‘set the tone’ by encouraging students to share ideas at the outset.

(b) ***Using Open-Ended Problem Solving as Differentiated Learning Tasks in regular mathematics teaching and learning sequences.*** For example, a one-hour mathematics lesson structure might be:

- Warm-up game / activity (5 minutes);
- Explicit teaching session (10 minutes);

- Activities and Exercises (Understanding and Fluency), followed by open-ended Problem Solving Task (Problem-solving & Reasoning) (up to 40 minutes); and
- Reflection / Review (5 minutes).

During the 'Open-Ended Problem-solving' session of the lesson sequence, students who have effectively coped with the understanding and fluency activities (eg. a small set of textbook exercises or a paired 'drill and check' activity) can move into an open-ended problem that aligns with the current topic. This task could be one or several 'opened up' questions based on some closed questions from the text; a teacher-developed stand-alone open-ended problem with multiple parts and solutions; or a rich task or mathematical investigation designed to be completed over a series of lessons or throughout the duration of the current topic unit.

Having more confident or able students working on open-ended problems for a good 20 minutes of each lesson on a regular basis also frees up the teacher to work with students who are still struggling with understanding and fluency with concept.

(c) ***Using Open-ended 'Rich tasks' and mathematical investigations as alternative assessment tasks.*** Teachers can set individual or small group open-ended or rich tasks in mathematics as in-class project work, aligned with a topic area or unit of work, that can then be used for student assessment as an alternative to traditional 'pen and paper' tests or exercise sets.

Teachers might dedicate entire lessons to working on a rich task as a self-paced unit of work; set aside one lesson a week within their mathematics timetable for working on rich tasks; or alternately set rich tasks or mathematical investigations / projects as homework or assignment work (the disadvantage of the latter suggestion is that students are unable to benefit from discussion or interact with the ideas of others).

Using rich tasks for the purposes of 'assessment as learning' is powerful because they will inevitably provide a range of student responses, including a chance for students to show all that they know about the relevant content. They also provide an opportunity for students to transfer knowledge from a known context to a less familiar one and help teachers to decide what specific help students may require in the relevant content areas (Clarke & Clarke 2002).

Developing appropriate marking rubrics is one issue teachers will need to address is using open-ended or rich tasks as alternative maths assessment. As mentioned earlier, these should ideally be based on the 'Working Mathematically' outcomes in the NSW Syllabus at appropriate Stage levels.

The Australian Association of Mathematics Teachers produces a book called' Mathematics Assessment for Learning: *Rich Tasks and Work Samples* (RRP $50.00) This book is for teachers seeking to inform their teaching by the use of tasks which can help to identify what their students know and can do in mathematics. It presents appropriate tasks across a broad range of grade levels and demonstrated understandings and possible assessment rubrics.

2. Creating your own open-ended questions and problems

Teachers can reasonably quickly and easily convert existing 'closed' questions and problems – such as those found in a textbook or on an online maths package – into open-ended problems or activities.

Several possible ways to do this are:

1. Remove some data or assumptions form an existing closed question or problem to 'open it up' to allowing several solutions;

2. Provide a worked solution to an existing closed question or problem *without providing the original question* and ask students to compose a problem that will fit; or

3. Have students undertake mathematics investigations in which they must apply the current topic focus (*eg. multiplication, fractions of a whole, area, data collection and graphing…*) in a real world context.

(a) Removing data or assumptions from an existing closed question or problem to 'open it up'.

Consider the following example of a closed mathematics problem:

> *"The kitchen bench is 62.4 centimetres deep. How far does it stick out from the cupboards, if they are 57.9 cm deep?"*
>
> When we remove some of the data and change the wording of the question accordingly, students are left with an open-ended problem to solve. In addition, we can add an exploratory 'layer' to the problem to increase the challenge level and so differentiate the problem for students who wish to go further:
>
> *"I am ordering a new benchtop for my kitchen. My kitchen cupboards are 57.9 cm deep.*
>
> *How deep should I make my new benchtop to ensure it covers the cupboard depth with a reasonable amount sticking out from the cupboards?*
>
> *How much room will there be in front of the microwave oven once I place it on the new benchtop?"*

Notice firstly that the given parameters of one of the variables has been removed to allow for student discretion (ie, the depth of the new benchtop).

Secondly, students will also need to decide whether the bench in question is against a wall (so there's only one overhanging 'lip') or whether it's an island bench (ie, an overhang on both sides). If they move into the 'microwave' part of the problem, they will also need to decide (or measure) the width of a suitably sized microwave. They will then need to factor all these assumptions into their calculations.

Lastly, notice that this problem has relevance in the real world – and that this relevance is enhanced by opening up the problem. Anyone who has ever renovated a kitchen (or whose parents have renovated a kitchen) would recognise this as a very realistic problem scenario.

(b) Providing a worked solution to an existing closed question or problem without providing the original question and asking students to compose a corresponding problem.

Consider this original question and worked solution:

> *"Mum put on the roast at 4:30 p.m. It finished cooking 2 hours later. What time did it finish, in 24-hour time?"*
>
> *Solution: 4:30 pm + 2 hours*
>
> *= 6:30 pm.*
>
> *6:30 pm expressed as 24 hour time: 1200 + 0630*
>
> *= 1830*

An open-ended problem can be created simply by providing students with the following from the above:

> *"The solution is the 24 hour time of '1830'.*
>
> *Write a 'time' addition or subtraction word problem that will fit this answer."*

The levels of sophistication and creativity of student responses to the above will obviously vary significantly, even within a single grade level class. In fact, this is exactly one of the big advantages of open-ended problem solving – it allows students to demonstrate the extent to which they are able to model numerical fluency, creativity, mathematical communication and reasoning in their endeavours.

(c) Composing maths investigations in which students apply a topic in a real world context.

This type of open-ended problem solving is very much project based and could be used over a series of lesson or for a mathematics project or assignment. In a sense, this involves setting a 'rich' or authentic task for students to complete.

Here is an example of an open-ended maths investigation teachers might set for students in a 'data and statistics' topic:

> *"Collect some data on a topic of your interest from at least 15 of your friends and family. Your teacher can help you choose a topic if you're really stuck.*
>
> *Explain how you collected and recorded your data and then create a suitable graph of the data you have collected.*
>
> *Can you calculate an average, a range and a median for the data you have collected?*
>
> *In a paragraph, explain how your data, the graph or the information you have collected might be used to answer a useful question or to solve a problem."*

Tasks such as this allow students to utilise the mathematics they have learned in a way that is directly relevant to their own experiences.

Again, the level of complexity and creativity of student responses will obviously vary significantly within any group – and again, this makes such tasks very useful for making discerning assessment observations on students' capacity to 'work mathematically'.

3. Some Sources of Open-Ended Problems, Maths Investigations and Rich ('Authentic') Tasks

There are a range of online and print materials upon which teachers can base their own 'portfolio' of open-ended problem solving resources. The following are a selection that we have collated at AMSI Schools, as a starting point for teachers seeking to begin using open-ended problem solving with their class. *It is important to note that many of these materials are subject to copyright and schools will need to adhere to Australian and international copyright rules and regulations with regard to their reproduction and use, where appropriate.*

- The N-Rich Mathematics website is a great source of open-ended problems (and a fantastic resource for maths teachers overall): www.nrich.maths.org.

- New Zealand Maths is also a great website for problem solving. The following link will take you to the grade-levelled "Problem Solving" section of the site: http://nzmaths.co.nz/level-4-problems ; many (not all) of these activities are open-ended.

- A good source of open-ended puzzles that are suitable for warm-ups is the booklet "Mathematical Challenges for Able Students": http://www.bgfl.org/bgfl/custom/files_uploaded/uploaded_resources/12212/mathspuzzlesall.pdf. *(Note that this is authored in the UK and so teachers use these activities they will need to change the 'pounds and pence' to 'dollars and cents').*

- Another source of open-ended group puzzles and problems for warm-ups and topic enrichment are the 'Group Challenge' sets A – E: http://www.stmaryscofe.org/?q=content/maths-challenge. *(Again, some of the tasks in these sets are closed problems while others have multiple solutions and are therefore open-ended).*

- The UK based National Centre for Excellence in the Teaching of Mathematics (NCETM) has produced a series called "Mastery Assessments" at each (UK) grade level from K through 6. These provide topic-by-topic exemplars of closed questions / problems, matched alongside so-called 'mastery questions (usually open-ended problems) in the same vein. https://www.ncetm.org.uk/resources/46689.

- Dr R Pelfrey (University of Kentucky) has authored a booklet called 'Open-Ended Problems for Mathematics', containing problem sets and solutions for Grades 4, 5 and 8 (US): https://www.uky.edu/OtherOrgs/ARSI/www.uky.edu/pub/arsi/openresponsequestions/mathorg.pdf.

- Finally, a terrific resource to have for group problem solving activities, many of which are open-ended, is Peter Gould's 'Cooperative Problem Solving in Mathematics', which contains teaching activities and accompanying blackline masters to engage student groups. The book is available from the Mathematical Association of NSW at: http://www.mansw.nsw.edu.au/shop/books-middle-school/co-operative-problem-solving.

Examples of 'Rich Tasks' and Mathematical Investigations

There are a range of rich tasks available on sites such as:

- the N-Rich Mathematics website (http://nrich.maths.org/6089);

- the NZ Maths site (http://nzmaths.co.nz/level-4-rich-learning-activities); and

- the AMSI Schools Website (http://calculate.org.au/category/curriclinks/).

In addition, the Victorian Department of Education has a series of rich mathematics tasks: http://www.education.vic.gov.au/school/teachers/teachingresources/discipline/maths/assessment/Pages/authtasks.aspx#5.

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