Wing Design

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These notes accompany the video on Wing Design.

Click on this link to watch the video: **Wing Design Video**

Introduction

It was only a little over 100 years ago that the human dream of flying like a bird became a reality. In order to realise that dream it was necessary to understand how to produce forces that would overcome gravity and air resistance, known as drag. This required equations for force, area and pressure. Once these were determined, a way needed to be found to construct vehicles that were sufficiently strong to withstand the forces without breaking under the stress. This was achieved not just by finding formulas but also by using calculus.

The Four Forces

There are four forces, acting in pairs, that make flight possible. The first two are **thrust** and **drag**. The second pair are **lift** and **gravity** or weight.

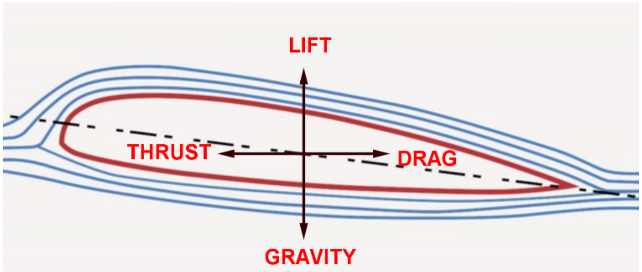


Figure 1: The Four Forces

In order to move around it is necessary to provide a push or pull against the ground, water or air. This push, or thrust, is resisted by the air or water in front of the moving object. When it comes to flight, the name given to this resistance is drag. When moving at a constant speed, thrust and drag are equal in size but opposite in direction. To speed up, or accelerate, thrust must be greater than drag; to slow down, drag must be greater than the thrust.

Gravity is what causes anything without support to fall to the ground. In order to counter the effect of gravity there must be some other force acting upward on the object. For balloons, this force is produced by making the density of the balloon less than that of the air surrounding it. The air essentially pushes the balloon higher into the air. This is called buoyancy and is also what allows ships made of steel to float on water.

For birds and planes it is not possible to change their density to be less than that of air so a different force, called **lift**, is used instead.

Lift

Lift is produced by the properties of the cross-sectional shape of the wings of birds and planes. This shape is known as the aerofoil, or airfoil in the US and Canada. As the wing moves forward through the air it splits the airstream into two parts, one going over the wing and the other going under the wing. The two different but complementary physical properties then combine to produce lift. First, two air streams flow at different velocities, creating a pressure differential between the upper and lower surfaces of the wing due to an effect explained by **Bernoulli's Principle**. Secondly, the aerofoil acts at an angle to deflect the flow of air downward. **Newton's laws of motion** then describe the generation of a reaction force pushing the wing, and thus the plane, upward.

For level flight, lift and weight balance each other out, having the same magnitude, or size, but acting in opposite directions. **Weight**, in mathematics and physics, is the force on an object produced by gravitational acceleration.

$$W = m.g$$

Exercise 1

A Boeing 787-9 Dreamliner (henceforth referred to as "a 787") has a maximum take-off mass of 251 metric tonnes. Determine the lift necessary to enable a 787 to take off.

The Lift Equation

Knowing the value of lift needed for flight is one thing. Constructing a craft that will actually fly is quite another. The formula for lift is:

$$\text{Lift} = \text{Lift Coefficient} \times \frac{\text{density} \times (\text{velocity})^2 \times \text{wing area}}{2}$$

or

$$L = C_L \frac{\rho v^2 A}{2}.$$

Below is a glossary of the terms involved in the equation.

Density, ρ, is air density at a given temperature and altitude.

Velocity, v, is the airspeed of the plane. In this equation it is only the magnitude of the velocity that is considered. The direction is understood to be forward. Airspeed is a result of thrust and winds through which the plane travels. A head wind (that is, wind that the plane is heading into) will add to the speed generated by the plane's engines. Tail winds, on the other hand, will subtract from what is generated by the engines.

Wing area, A, is the sum of the plan view areas (also known as cross-sectional areas) of both wings and some component resulting from the area of the fuselage or body of the plane.

Velocity and wing area can be easily calculated. Air density needs to be measured and tables of typical values have been constructed since before powered flight first began with the Wright brothers.

The **Lift Coefficient**, C_L, on the other hand, is a complicated variable that takes into account the variety of factors including aircraft design, the properties of air as a fluid, the angle of the wing to the airstream and so on. As a result, lift coefficients are usually determined experimentally for each aircraft design and for different air densities which decrease as altitude increases. Lift coefficients typically fall between 0.1 and 2.

Wing Area

Even though the secret to flight is the shape of the aerofoil, lift is calculated by treating the wing as a two dimensional shape. A plan view of one 787 wing is shown below, along with coordinate values of each of the vertices. All measurements are in metres.

Exercise 2

a Calculate the area of one 787 wing.

b The fuselage, as well as the wings, provides some lift. Using the value of 25.98 m^2 for the component of area provided by the fuselage, calculate the value of the total wing area of a 787.

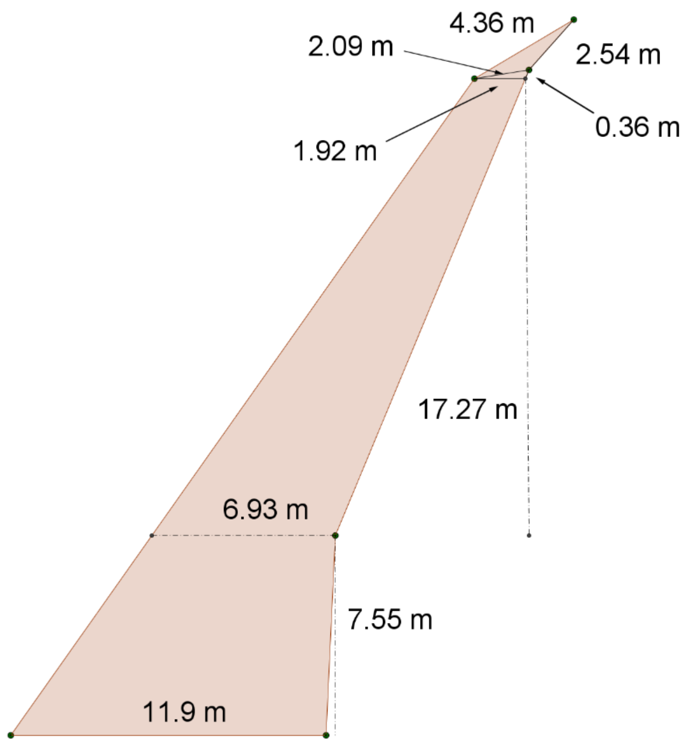


Figure 2: Wing dimensions for Boeing 787 Dreamliner

Velocity

For the purpose of this exercise, let take-off velocity for the 787 be 300 km h^{-1} and its cruising velocity be 913 km h^{-1}.

Altitude (m)	Air Density, ρ, (kg m^{-3})
0	1.2250
1000	1.1120
2000	1.0070
3000	0.9093
4000	0.8194
5000	0.7364
6000	0.6601
7000	0.5900
8000	0.5258
9000	0.4671
10000	0.4135

Table 1: Density table adapted from
`http://www.engineeringtoolbox.com/standard-atmosphere-d_604.html`

Exercise 3

a Use the table of density values with your answers to Exercises 1 and 2 to determine the value of the lift coefficient for a 787 at cruising velocity at 10 000 m.

b Similarly, calculate the lift coefficient for a 787 at take-off.

c What assumption did you make, and will this always be true?

There are online lift coefficient calculators that you can use to check your answer if you wish. One of these is given in the References at the end of this module.

Exercise 4

The introductory video mentions that the 787 Dreamliner's Moveable Trailing Edge (MTE) changes wing area to provide more lift at lower speeds. Explain why this is so with reference to the lift equation. For purposes of simplification, you may take it that the MTE adds an extra metre to the two major sections of the trailing edge of each wing when fully extended. Also for the purposes of simplification, you may use the value for the lift coefficient C_L calculated in Exercise 3.

Drag

The equation for drag is very similar to the one for lift.

$$\text{Drag} = \text{Drag Coefficient} \times \frac{\text{density} \times (\text{velocity})^2 \times \text{wing reference area}}{2}$$

or

$$D = C_D \frac{\rho v^2 A}{2}$$

At a first glance, all of the terms appear to be the same, with the exception of the subscript of the coefficient. Looking more deeply, however, the area used is called the wing reference area, whereas for lift it was the cross-sectional wing area. This is a significant difference. The causes for drag are more varied than those for lift and, as a result, the determination of the area to use is more complicated.

The simplest depiction of drag is the friction between the air and the skin of an aircraft. In such a case, then, the area to use would just be the total surface area of the plane. This is called **skin friction drag**. As the Wright brothers discovered, however, craft with the same area can have significantly different amounts of drag depending on how they are shaped. Wing span and chord length (see page 12) play a particularly important role in this. Drag produced by the shape of the plane is known as **form drag**.

A third component to drag occurs as a result of lift acting on the wings, especially at the tips and edges. As the two air currents separate and recombine the resultant swirling induces drag on the wing. Sensibly enough, this is called **induced drag**.

The fourth and final major cause of drag is called **wave drag** and is produced by shock waves for craft as they reach and then go beyond the speed of sound. All four of these sources need to be considered when designing aircraft.

In practice, when choosing a reference area, designers may take the surface area, the frontal aspect area (which is the area of the plane moving at right angles to the airflow), or the cross-sectional wing area if they want to compare drag directly with lift. When it comes to determining the drag coefficient they then use a wind tunnel to simulate all the possible conditions that a plane may encounter.

A current area of research for Boeing and other aeronautical companies is how different textures on the skin of a plane affects drag. Several aircraft manufacturers have been exploring the use of textured paint and materials to reduce skin friction drag.

Exercise 5

The 787 has a cruising speed of 913 km h^{-1} (equivalent to 253 ms^{-1}), at around 10 000m. Most passenger aircraft cruise at a thrust output of 85% of maximum. For a 787, the maximum thrust is 320 kN.

a Use this information to determine the drag coefficient for a 787 at cruising speed. Remember that standard atmospheric densities were provided in the table on page 8.

b Explain why this value for the drag coefficient can only be considered a theoretical estimate.

Acceleration and Climbing

When a plane is travelling horizontally at a constant velocity, all the forces balance out in pairs:

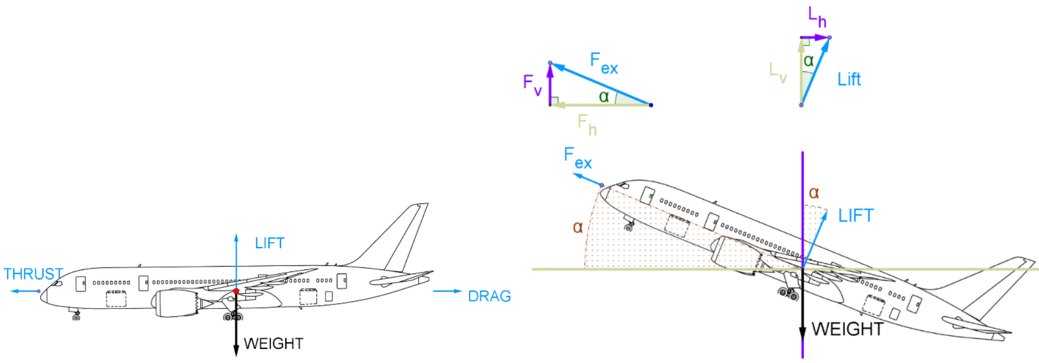
$$\text{Thrust} = -\text{Drag}$$

$$\text{Lift} = -\text{Weight}.$$

When the plane accelerates horizontally, thrust is greater than drag and **excess thrust**, F_{ex}, is

$$F_{ex} = ma = \text{Thrust} - |\text{Drag}|$$

In a climb, weight continues to act vertically downward but thrust and lift are rectified into vertical and horizontal components.



(a) Plane in horizontal flight

(b) Plane in a climb

Exercise 6

The Excess Thrust equation can also be written as:

$$F_{ex} = ma = \text{Thrust} + \text{Drag}$$

a Identify the differences in the two equations and explain why they are equivalent.

b Write equations for the vertical and horizontal components of ma in a climb.

Aerofoils

As mentioned above, the secret to flight is the cross sectional shape of the wing called the aerofoil. This is typically rounded and thick at the front, or leading, edge, and tapered to a point at the back, or trailing, edge. Yet, an examination of birds and bats shows that their wings are not all identical in shape or size. These variations allow for different qualities such as speed, manoeuvrability and hovering.

For the first decade or so of human powered flight from 1903, the development of aerofoil shapes was largely a matter of past experience and experimentation. This began to change in the years leading up to the First World War and after. In the United States, for example, the National Advisory Committee for Aeronautics (NACA) was founded in 1915 with the express purpose of developing a more systematic approach to aerofoil design. Over the years from 1915 to 1958 NACA developed several series of aerofoils that bear their names. Of these, the NACA 4-digit was the first, and the mathematics behind it is very accessible to secondary level students.

Historical note: in 1958 the NACA was disbanded and its assets and facilities transferred to the newly established National Aeronautics and Space Administration, NASA.

The NACA 4-Digit Series

There are three distinct phases in the development of the NACA 4-digit aerofoils:

1. The establishment of the mean camber line;
2. Calculation of the aerofoil thickness distribution;
3. Computation of the final coordinates of the aerofoil surfaces.

While computers now make these calculations routine and almost trivial, it is worth noting that for the entire lifetime of the NACA they were conducted by hand. Even in the

1960s with space flight and trips to the Moon, the computing power at NASA was less than what can now be found in a typical mobile phone.

Describing a NACA 4-digit aerofoil

The 4 digits in a NACA 4-digit aerofoil relate to three parameters used to describe the shape of the aerofoil: m, p and t. To describe these parameters, we first have to understand the concepts of camber and chord.

Camber is a term used in many areas of study to describe the amount of curvature or angular positioning between elements in a design. In wing design it means the asymmetry between the top and bottom surfaces of an aerofoil. In the top example shown in Figure 4, the NACA 0016 is symmetrical so there is no camber. By comparison, the bottom example – the NACA 6412 aerofoil – is distinctly asymmetric so there is a lot of camber.

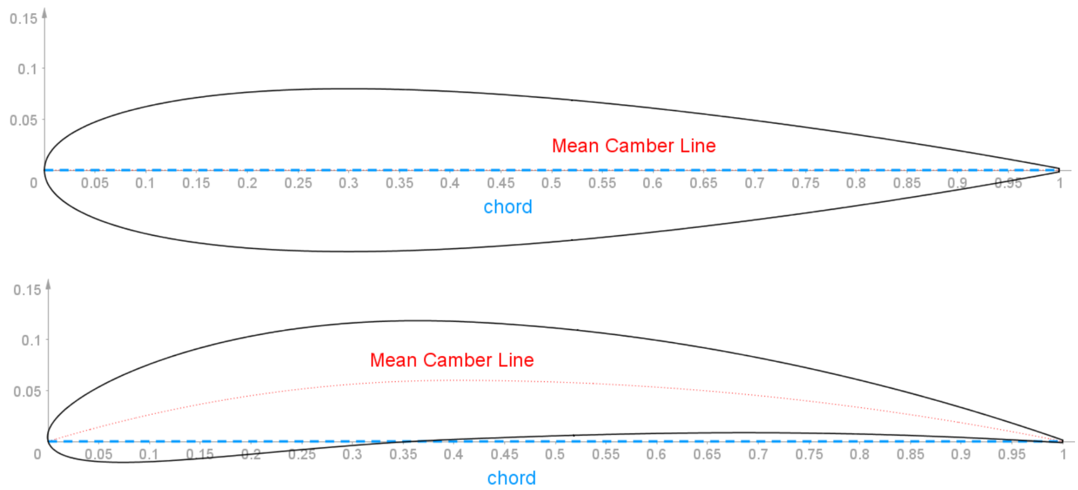


Figure 4: Comparison of NACA Aerofoils 0016 and 6412

The **chord** of an aerofoil is the straight line joining the leading and trailing edges of the aerofoil. These are indicated in blue in Figure 4. The *chord length* is the length of this line. To make computations easier, the chord length is usually standardised to be equal to 1.

The mean camber line is the curve consisting of all the points halfway between the top and bottom surfaces of the aerofoil. These are indicated in red in Figure 4. If there is no camber, the chord and mean camber line coincide. The further away the red line is from the blue, the greater the camber.

To describe the exact shape of an aerofoil, we need to know the maximum camber of the wing, how far along the chord the maximum occurs, and the maximum thickness of the wing. These three parameters are labelled m, p and t and are described as follows:

- m, the maximum camber, is the maximum distance between the chord and mean camber line, as a percentage of chord length. It is the percentage asymmetry between the upper and lower surfaces. The higher the value of m, the more asymmetric the wing is.
- p, the position of the maximum camber, is a value between 0 and 1 and indicates the distance along the chord from the leading edge of the aerofoil where the maximum camber occurs.
- t, the maximum thickness of the aerofoil, is the maximum distance between the upper and lower surfaces, measured as a percentage of the chord length.

The naming convention for the NACA 4-digit series comes from the values of m, p and t. For example:

- The NACA 9315 aerofoil has a maximum camber, m, of 9% which is located at a point p which is 0.3 chord lengths from the leading edge, and it has a maximum thickness t which is 15% of the chord length.
- The NACA 6412, as shown again in Figure 5 with parameters m and p indicated, has a maximum camber of 6% located 0.4 chord lengths from the leading edge, and with a maximum thickness which is 12% of the chord length.

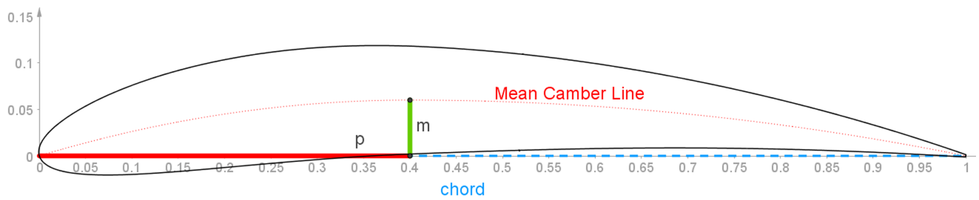


Figure 5: NACA Aerofoil 6412 showing parameters m and p

Exercise 7

Describe the parameters of the following aerofoils:

a Cessna models using NACA 2412

b Boeing 500F helicopter using NACA 0012.

The shape of a NACA 4-digit aerofoil can be precisely described using the parameters m, p and t. The next three sections will go into this in more detail.

The Mean Camber Line

For each value of x along the length of the chord, the y-coordinate of the mean camber line y_c is calculated by the equation:

$$y_c(x) = \begin{cases} \dfrac{m}{p^2}\left(2px - x^2\right), & \text{for } 0 \le x \le p \\ \dfrac{m}{(1-p)^2}\left[(1-2p) + 2px - x^2\right], & \text{for } p < x \le 1. \end{cases}$$

Exercise 8

a Show that the two sections of the mean camber line always form a smooth join when $x = p$.

b The two components of the function are both based on $(2px - x^2)$. Explain how the additional terms, $\frac{m}{p^2}$, $\frac{m}{(1-p)^2}$ and $(1-2p)$ act to transform the same parent function into each section.

Aerofoil Thickness

For each value of x along the length of the chord, the thickness of the aerofoil both above and below the mean camber line is calculated by the equation:

$$y_t(x) = \frac{t}{0.2}\left(0.2969\sqrt{x} - 0.1260x - 0.3516x^2 + 0.2843x^3 - 0.1015x^4\right) \tag{1}$$

That is, $y_t(x)$ is the half-thickness of the aerofoil at a distance of x from the leading edge.

Aerofoil Surface Coordinates

The hybrid function for camber, y_c, and the function for aerofoil thickness, y_t, are now used in combination to determine a set of locus points along the surface of the aerofoil. The upper surface has coordinates (x_U, y_U) while the lower surface has coordinates (x_L, y_L). These are given by:

Upper Surface	Lower Surface
$x_U = x - y_t \sin\theta$	$x_L = x + y_t \sin\theta$
$y_U = y_c + y_t \cos\theta$	$y_L = y_c - y_t \cos\theta$

Notice that the x and y coordinates for both the upper and lower surfaces are calculated using functions dependent on y_c and y_t.

The angle, θ, is the angle of inclination at each point along the mean camber line and is found using

$$\theta = \arctan \frac{dy_c}{dx}. \tag{2}$$

For the NACA 6412 aerofoil, when $x = 0.2091$, then the angle of incidence is $\theta = 8.1467°$ and the coordinates of the mean camber lines and upper and lower surfaces are shown below in Figure 6.

You can explore the way in which the NACA 4-digit series aerofoils change shape according to the parameters m, p and t by opening the online geogebra file. Clicking on Figure 6 will launch an interactive version in your browser.

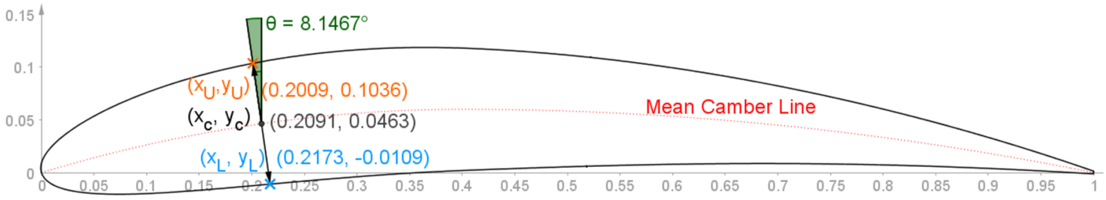


Figure 6: NACA 6412 aerofoil angles of incidence

It is worth noting again that all of the calculations required for NACA series aerofoils were performed manually to a precision of four decimal places.

Exercise 9

a Use the parameters for a Cessna light plane from Exercise 7 to find the angle of the mean camber line for a value of $x = 0.1$.

b Use this angle to determine the coordinates for the upper and lower surfaces again for $x = 0.1$.

Stress

Imagine a gymnast performing the rings. At one stage in the routine they hold their body horizontal to the ground with their arms out to the side, as in Figure 7. The gymnast's arms and shoulders are supporting their entire weight. This is exactly the same for planes except that instead of holding onto rings it is just a difference in pressure that keeps the plane in the air.

Figure 7: Forces on a gymnast are similar to those on the wings of a plane

The wings of a plane are some of the most technically complex products of engineering ever developed. As well as having a shape that is accurate to within a quarter of a millimetre, each wing must be able to withstand the forces placed on them from every direction.

A diagram of the internal structure of a wing (see Figure 8) shows that it is the **ribs** that provide the wing with its aerofoil shape. The ribs themselves are connected together by **spars**. It is the spars that give strength to the wings and which have to tolerate the major stresses involved in flight.

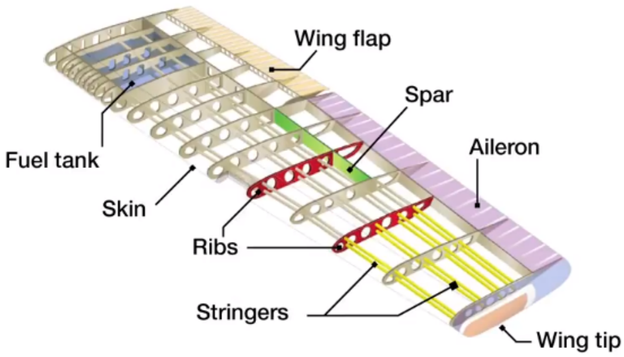


Figure 8: The internal structure of a wing

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Distributed load

In this section, and those following, the variable x refers to the distance along the wing, measured from the point where the wing joins the body of the plane (fuselage).

Wing spars can be modelled as simple beams. The force on each wing is half the weight of the aircraft plus fuel, cargo, passengers and crew. For a 787 Dreamliner this (force on each wing) can be as much as 125 tonnes. The lift force is distributed along the length of each of the wings and can be calculated simply by:

$$\omega(x) = -\frac{\text{Force}}{\text{wing length}} \tag{3}$$

ω is known as the **distributed lift load** of a wing. Although it is written as a function of x, it is constant along the length of the wing.

Exercise 10

a Calculate the distributed lift load for a 787 Dreamliner given a total mass of 220 tonnes and a wing length of 30 m. For the purpose of simplification take g, the acceleration due to gravity, to be 10 ms^{-1}.

b Explain why your answer is a negative value.

Shear Force

The shear force on a wing is the force acting in the beam perpendicular to the x-axis. As the name suggests, if the shear force is strong enough then it will cause the material of the wing to shear or rip apart. Lift is forcing the wings up while gravity is pulling the fuselage down. This creates tension, or stretching forces, in the underside of the wing and compression in the top of the wing. This can be modelled by placing a ruler on two blocks at either end and then pushing down on its middle.

Shear force, V, can be calculated by integrating the equation for distributed load, ω.

That is,

$$V(x) = -\int \omega(x)dx.$$

Performing the integration above, and remembering that $\omega(x)$ is constant, gives:

$$V(x) = -\omega x + C.$$

In the video, the shear force for the Dreamliner is given as $V(x) = 36667x - 1100000$.

To arrive at a C value of 1 100 000 it is necessary to realise that $V(x)$ is at its minimum value (that is, zero) when $x = 30$ (i.e. at the wing tip). So, the value of C is found by:

$$\begin{aligned} C &= V(x) + \omega x \\ &= 0 + (-36667) \times 30 \\ &= -1100000. \end{aligned}$$

Exercise 11

The ability of the beam to resist shear force is important. Explain why. Use a diagram.

Bending Moment

Materials experiencing shear forces, where one end of the object is fixed, want to try and bend in response to the force. This is analogous to the torque on a nut produced by a spanner. The difference is that we do not want anything to turn. This bending moment is highest at the fixed point junction between the wing and the fuselage.

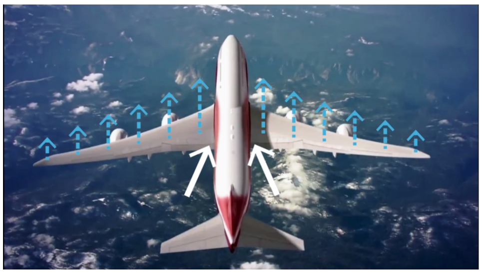

Figure 9: Shear force on the wing of a 787 Dreamliner

Bending moment is the integral of shear force:

$$M(x) = \int V(x)dx$$

Exercise 12

Use calculus to verify the bending moment equation for the 787 given in the video (at 3:37 minutes). Remember to consider an appropriate boundary condition.

Double Integrals

The video next proceeds to discuss the **second moment of area** and introduces the concept and nomenclature for double integrals. While this is beyond the scope of any Year 12 mathematics course in Australia, it is an interesting extension that takes only a short amount of time. We will begin this section by first discussing displacement, velocity and acceleration and then proceed to use the derivation of bending moment from distributed lift load as an example of the use of double integrals.

Displacement, **velocity** and **acceleration** are all vector quantities relating to the motion of an object. In fact, Newton and Leibniz discovered that velocity is the derivative of displacement, and acceleration is the derivative of velocity. This also makes acceleration the second derivative of displacement.

An object experiencing acceleration will have a displacement equation in the form of a quadratic or higher polynomial. For example:

$$X(t) = at^2 + bt + c$$

Velocity, $V(t)$, is then:

$$\begin{aligned} V(t) &= \frac{dX}{dt} \\ &= 2at + b. \end{aligned}$$

In turn, acceleration, $A(t)$ is:

$$\begin{aligned} A(t) &= \frac{dV}{dt} \\ &= 2a. \end{aligned}$$

A shorthand way of writing A directly in terms of X is to use the second derivative notation

$$A(t) = \frac{d^2X}{dt^2}.$$

Now, since integration is the inverse process to differentiation, it holds that a shorthand notation for repeated integration also exists. In this case the integrals are nested inside one another. After each integration the constant is found by using the boundary conditions.

With acceleration, velocity and displacement, the process goes as follows. We start with a formula for acceleration,

$$A(t) = 9.8$$

and integrate this to find an expression for velocity:

$$V(t) = \int A\,dt = At + b,$$

where b is a constant of integration.

If the object starts moving from rest, then $V(0) = 0$, so $b = 0$. However, if the object is already moving before the force is applied then $b = u$, where u is the initial velocity.

So we can write

$$V(t) = At + u = 9.8t + u.$$

To find the displacement, X, we now integrate the velocity V:

$$X(t) = \int V(t)\,dt = \int (9.8t + u)\,dt = \frac{9.8}{2}t^2 + ut + c$$

where c is a constant of integration.

If displacement is 0 when $t = 0$, then $c = 0$, giving us

$$X(t) = \frac{9.8}{2}t^2 + ut.$$

This is identical to Newton's equation of motion for displacement under constant acceleration. We can also write X in terms of a double integral of the acceleration:

$$X(t) = \int \left(\int A\,dt \right) dt.$$

In this example, both integrations are with respect to t. It is possible, however, for the integrating variable to be different at each stage, as we will soon see.

Exercise 13

Express the bending moment, $M(x)$, as the double integral of distributed load, $\omega(x)$.

Second Moment of Area, I

This is the property of a two dimensional shape that relates to how much it is deflected when experiencing a load. The cross section being considered for this in a wing is the vertical face of the spar, which we remember is modelled by a rectangular beam.

Figure 10 shows diagrams of the same beam placed in two different orientations. When a load is placed on top of the beam, the orientation on the left will bend less than the one on the right. We will see that this is confirmed by the calculation of the second moment of area, I, which shows that the left beam has a higher value for I than the right one. A higher value of I means a stronger beam.

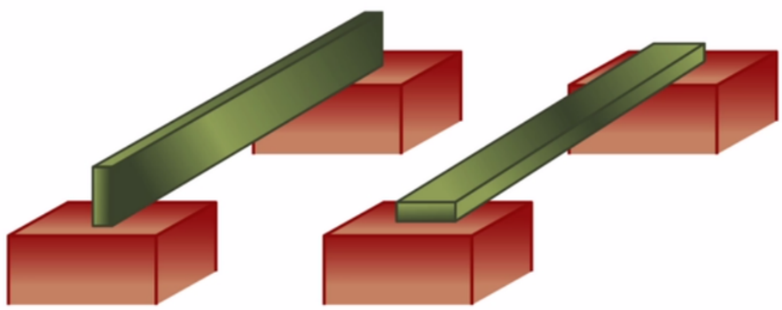

Figure 10: A beam in two different orientations

Calculating I

The second moment of area measures the distribution of points in a shape around a particular axis. If the shape has many points that are far away from the given axis, this will create a higher value for I. We will look at how the calculation works in the case of a rectangular beam.

Rectangular beams have dimensions of height, h, in the y-axis, and breadth, b, in the x-axis. Calculating I about the x-axis involves the use of a double integral. The first integration is in the y-axis and the second is in the x-axis. We assume that the beam is centred at $(0,0)$ as in Figure 11. This means that integration along the y-axis happens over the interval $(-\frac{h}{2}, \frac{h}{2})$, and integration along the x-axis happens over the interval $(-\frac{b}{2}, \frac{b}{2})$.

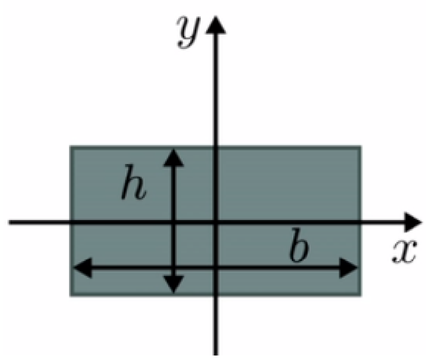

Figure 11: Beam with height h and breadth b

Thus the second moment of area of a rectangular beam about the x-axis is calculated as

$$I_x (= I_{xx}) = \int_{\frac{-b}{2}}^{\frac{b}{2}} \left(\int_{\frac{-h}{2}}^{\frac{h}{2}} y^2 dy \right) dx.$$

Performing the integration produces the result shown in the video:

$$I_x = \frac{bh^3}{12}$$

Note: the second moment of area is also referred to in the video as the **moment of inertia**. While this is acceptable for the purposes of the current discussion, it is worth pointing out that there are differences between the two quantities when conducting a more in-depth analysis.

Exercise 14

a Using values of $b = 2$ and $h = 5$, and then conversely, $b = 5$ and $h = 2$, calculate I_x in each case and verify that, for these values of h and b, the value of I_x is greater when $h > b$ than when $h < b$.

b Is it always true that I_x is larger when $h > b$ than if $h < b$, no matter what values are chosen for h and b? Justify your answer with an argument or counterexample.

Bending Stress

The final section of the video discusses the calculation of bending stress at any point along the x-axis of the wing. This **bending stress equation** is:

$$\sigma_x(y) = \frac{M_z y}{I_x},$$

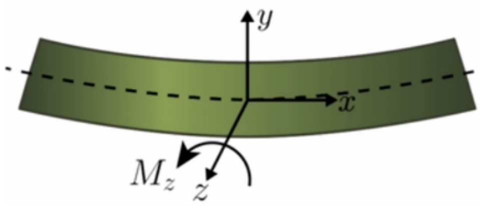

Figure 12: Bending Moment

where M_z is the **bending moment about the z-axis**, I_x is the **second moment of area along the x-axis** and y is the perpendicular distance from the point to the x-axis.

Looking carefully at the diagram for the bending moment in Figure 12, it is shown as a rotation about the z-axis, which is perpendicular to the face of the beam pointing outward.

Exercise 15

a Calculate σ when $x = 15$, $y = 0.5$, $b = 30$ and $h = 0.5$.

b What do the values of x, y, b and h refer to in terms of a 787's wing?

Conclusion

All of the mathematics explored in this activity is accessible to students studying a calculus course at Year 12. Indeed, high school algebra and calculus is fundamental to the history, development and realisation of the dream of human flight. Science and engineering courses at university build on this foundation so that, in under one hundred years from the first short flight at Kitty Hawk, air travel is now commonplace and, moreover, the safest known form of transport. It is only by continuing to produce qualified engineers who understand the mathematics of all aspects of powered flight that humanity will continue to reach for the skies.

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Solutions to Exercises

Exercise 1

We have $m = 251000$ kg, $g = 9.8$ ms^{-2}. The lift necessary to counteract the weight of the plane is therefore

$$W = 251000 \times 9.8 = 2459800\text{N},$$

i.e. roughly 2.5 million Newtons.

Exercise 2

a To calculate the total area of the wing, we split the wing into two trapezia and two triangles. These areas, from bottom to top, will be called A_{trap1}, A_{trap2}, $A_{\text{triangle1}}$ and $A_{\text{triangle2}}$.

The area of a trapezium of height h and sides (perpendicular to h) of length a and b is

$$A = \frac{a+b}{2}h.$$

Applying this to our two trapezia gives us:

$$A_{\text{trap1}} = \frac{11.9 + 6.93}{2} \times 7.55 \;=\; 71.08 \text{ m}^2$$

and

$$A_{\text{trap2}} = \frac{6.93 + 1.92}{2} \times 17.27 \;= = 76.42 \text{ m}^2.$$

The area of a triangle with side lengths a, b and c can be calculated from Heron's formula

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

where $s = \frac{a+b+c}{2}$, i.e. half the perimeter of the triangle.

For Triangle 1 we have

$$s_1 = \frac{1.92 + 0.36 + 2.09}{2} \;=\; 2.185 \text{ m}$$

so we have

$$A_{\text{triangle1}} = \sqrt{s_1(s_1 - 1.92)(s_1 - 0.36)(s_1 - 2.09)} \;=\; \sqrt{0.1004} \;=\; 0.317 \text{ m}^2.$$

For Triangle 2 we have

$$s_2 = \frac{2.09 + 4.36 + 2.54}{2} \;=\; 4.495 \text{ m}$$

so we have

$$A_{\text{triangle2}} = \sqrt{s_2(s_2 - 2.09)(s_2 - 4.36)(s_2 - 2.54)} \;=\; \sqrt{2.853} \;=\; 1.69 \text{ m}^2.$$

The total area of the wing is then

$$\begin{aligned} \text{Total Area} &= 71.08 + 76.42 + 0.317 + 1.69 \\ &= 149.51 \text{ m}^2. \end{aligned}$$

b The total wing area is double the area of one wing plus the area of the fuselage. This gives

$$\begin{aligned} \text{Wing Area} &= (2 \times 149.51) + 25.98 \\ &= 325 \text{ m}^2. \end{aligned}$$

Exercise 3

a First transpose the lift equation to make C_L the subject.

$$L = C_L \frac{\rho v^2 A}{2} \Rightarrow C_L = \frac{2L}{\rho v^2 A}.$$

At cruising velocity we have $v = 913$ km h^{-1}, $A = 325$ m^2, $\rho = 0.4135$ and $L = W < 2459800$ N. Before we can calculate C_L we need to convert the velocity into SI units (i.e. ms^{-1}) :

$$v = \frac{913000}{3600} = 253.61 \text{ ms}^{-1}.$$

So,

$$C_L = \frac{2 \times 2459800}{0.4135 \times (253.61)^2 \times 325}$$
$$= 0.5692.$$

At a near empty fuel weight of 180 metric tonnes this changes to $W = 1764000$ N, which gives

$$C_L = \frac{2 \times 1764000}{0.4135 \times (253.61)^2 \times 325}$$
$$= 0.4082.$$

Note that this provides a range of figures. If other values for weight are used, they should be accompanied by justifications.

b At take-off we have $v = 300$ km h^{-1}, $A = 325$ m^2, $\rho = 1.2250$, $L = W \approx 2459800$ N. Converting the velocity to ms^{-1} gives us:

$$v = \frac{300}{3.6} = 83.33 \text{ ms}^{-1}.$$

So,

$$C_L = \frac{2 \times 2459800}{1.2250 \times (83.33)^2 \times 325}$$
$$= 1.7795.$$

c This question is designed to promote discussion. Most people will use the 1.2250 value for air density, which is the value at sea level. However, not all airports are at sea level. Some are actually above 4000 m and so this changes the results significantly.

Exercise 4

When the moveable trailing edge is extended, the area of the wing is increased by the area of rectangles of width 1m added to the trailing edge of each of the two trapezia. This additional area is calculated as:

$$\begin{aligned}\text{Added Area} &= (7.55 \times 1) + (17.27 \times 1)\\ &= 24.83 \text{ m}^2.\end{aligned}$$

The new total wing area is then

$$\begin{aligned}\text{New Total Area} &= 325 + (2 \times 24.83)\\ &= 374.66 \text{ m}^2.\end{aligned}$$

The area is the only parameter to have changed so we still have $v = 83.33\text{ms}^{-1}$, $\rho = 1.2250$ and $C_L = 1.7795$. This gives the lift as

$$\begin{aligned}L &= C_L \frac{\rho v^2 A}{2}\\ &= 2835594 \text{ N}.\end{aligned}$$

This is an increase of 375 794 N.

Exercise 5

a We first need to calculate the drag of the 787. If the plane is at cruising speed, we can assume that thrust and drag are equal in magnitude (and opposite in sign). The exercise tells us that that the plane has a thrust output of 85% of its maximum value of 320 kN, giving us

$$D = -0.85 \times 320000 = -272000 = -272 \text{ kN}.$$

We know that

$$D = C_D \frac{\rho v^2 A}{2}$$

so we can find C_D by rearranging this equation to make C_D the subject:

$$C_D = \frac{2D}{\rho v^2 A}.$$

To perform this calculation it is necessary to select a reference area. For comparison with lift, the cross-sectional area of the wing is appropriate, so we will use $A = 325\text{m}^2$. The exercise tells us that our speed is $v = 913 \text{ km h}^{-1} = 253 \text{ ms}^{-1}$, and Table 1 tells us that $\rho = 0.4135 \text{ kg m}^{-3}$. Thus,

$$\begin{aligned}C_D &= \frac{2 \times (-272000)}{0.4135 \times 253^2 \times 325}\\ &= -0.06324.\end{aligned}$$

b Firstly, the choice of reference area will change the calculated value of the coefficient. Also, the properties of the aircraft skin, the viscosity of the air and other atmospheric properties will all change actual drag. Temperature and humidity, for example, are significant factors. These will change the amount of thrust needed to keep the plane at a constant speed.

Exercise 6

a The first equation subtracts the absolute, or scalar, value of the drag force from thrust. The second equation adds the vector value of drag. Since drag acts backward to the plane's direction of motion, this has a negative value.

b The total vertical force on the aeroplane is the vertical component of the excess thrust, plus the vertical component of the lift, minus the weight:

$$\begin{aligned}(ma)_v &= F_v + L_v - W \\ &= F_{\text{ex}} \sin(\alpha) + L\cos(\alpha) - W.\end{aligned}$$

The total horizontal force on the aeroplane is the horizontal component of the excess thrust, minus the horizontal component of the lift:

$$\begin{aligned}(ma)_h &= F_h - L_h \\ &= F_{\text{ex}} \cos(\alpha) - L\sin(\alpha).\end{aligned}$$

Exercise 7

a A Cessna wing has a maximum camber, m, of 2% of the chord length, located at a point p which is 0.4 chord lengths from the leading edge and with a maximum thickness, t, of 12% of the chord length.

b A Boeing 500F helicopter rotor has a maximum camber, m, of 0% along the entire chord length (i.e. it is symmetrical) and it has a maximum thickness, t, of 12% of the chord length.

Exercise 8

a We first need to check that the function is continuous at the point $x = p$; i.e. that the equations for the two sections of the mean camber line agree at $x = p$.

For the front part of the line, we have

$$y_c(p) = \frac{m}{p^2}(2p^2 - p^2) = \frac{mp^2}{p^2} = m.$$

For the end part of the line, we have

$$y_c(p) = \frac{m}{(1-p)^2}(1 - 2p + 2p^2 - p^2) = \frac{m(1-p)^2}{(1-p)^2} = m.$$

Therefore the function is continuous at the point $x = p$. Next we need to check that it is smooth, i.e. that the derivative of the two functions agree at the point $x = p$.

Differentiating the front part of the line we get:

$$\frac{dy_c}{dx} = \frac{m}{p^2}(2p - 2x)$$

and differentiating the end part of the line we get:

$$\frac{dy_c}{dx} = \frac{m}{(1-p)^2}(2p - 2x).$$

When $x = p$, the gradient is zero in both cases, telling us that the join is smooth.

b The factors $\frac{m}{p^2}$ and $\frac{m}{(1-p)^2}$ dilate the parent function away from the y-axis (or along the x-axis). The term $(1 - 2p)$ translates the parent function this many units parallel to the y-axis.

Exercise 9

a A Cessna light plane is a NACA 2412 aerofoil, so $m = 0.02$ and $p = 0.4$. We want to find the angle of the mean camber line at $x = 0.1$. Since $x < p$, this is a point in the front section of the mean camber line, so the derivative is

$$\frac{dy_c}{dx} = \frac{m}{p^2}(2p - 2x).$$

Substituting for m, p and x gives:

$$\frac{dy_c}{dx}(0.1) = \frac{0.02}{0.4^2}((2 \times 0.4) - (2 \times 0.1)) = 0.075.$$

Now we can calculate the angle of the mean camber line using Equation 2:

$$\begin{aligned}\theta &= \arctan\left(\frac{dy_c}{dx}\right)\\ &= \arctan(0.075)\\ &= 4.289^\circ.\end{aligned}$$

b To calculate the coordinates of the upper and lower surfaces of the wing, we first need to find the aerofoil thickness y_t at the point $x = 0.1$. Using Equation 1 and $t = 0.12$ we get:

$$\begin{aligned}y_t(0.1) &= \frac{t}{0.2}(0.2969\sqrt{x} - 0.1260x - 0.3516x^2 + 0.2843x^3 - 0.1015x^4)\\ &= \frac{0.12}{0.2}(0.2969\sqrt{0.1} - 0.1260(0.1) - 0.3516(0.1)^2 + 0.2843(0.1)^3 - 0.1015(0.1)^4)\\ &= 0.05.\end{aligned}$$

We also need the y-coordinate of the mean camber line:

$$\begin{aligned} y_c(0.1) &= \frac{0.02}{0.4^2}(2 \times 0.4 \times 0.1 - 0.1^2) \\ &= 0.00875. \end{aligned}$$

With our knowledge of x, y_t, y_c and θ we can now calculate the coordinates of the upper surface:

$$x_U = x - y_t \sin\theta = 0.1 - 0.05\sin(4.289^\circ) = 0.096$$

$$y_U = y_c + y_t \cos\theta = 0.00875 + 0.05\cos(4.289^\circ) = 0.0586$$

and the lower surface:

$$x_L = x + y_t \sin\theta = 0.1 + 0.05\sin(4.289^\circ) = 0.104$$

$$y_L = y_c - y_t \cos\theta = 0.00875 - 0.05\cos(4.289^\circ) = -0.0411.$$

So, at $x = 0.1$, the upper surface coordinates are $(0.096, 0.0586)$ and the lower surface coordinates are $(0.104, -0.0411)$.

Exercise 10

a Using Equation 3 and calculating the force as ma where $m = 110$ tonnes (assuming a wing carries half the mass of the plane) and $a = 10\ \mathrm{ms}^{-1}$ gives us:

$$\begin{aligned} \omega(x) &= -\frac{110000 \times 10}{30} \\ &= -36666.67\ \mathrm{Nm}^{-1}. \end{aligned}$$

b The lift load is a force due to the weight of the plane pulling the wing downwards, hence the negative sign.

Exercise 11

The higher the tensile strength of the material (i.e. the stronger it is) the higher the force needed to cause it to rip or tear.

Exercise 12

We know that the shear force for the 787 Dreamliner is $V(x) = 36667x - 1100000$. This means we have

$$\begin{aligned} M(x) &= \int V(x)dx \\ &= \int (36667x - 1100000)dx \\ &= \frac{36667x^2}{2} - 1100000x + C. \end{aligned}$$

When $x = 30$, $M(x) = 0$ since there is no bending moment at the wing tip. Substituting gives $C = 16500000$ (with rounding). This corresponds to the values given in the video.

Exercise 13

$$M(x) = \int\int \omega(x)dx = \int\int -36667dx.$$

Exercise 14

a For $b = 2$ and $h = 5$ we have

$$\begin{aligned} I_x &= \frac{bh^3}{12} \\ &= \frac{2 \times 5^3}{12} \\ &= \frac{125}{6} = 20.83 \text{ m}^4. \end{aligned}$$

For $b = 5$ and $h = 2$ we have

$$\begin{aligned} I_x &= \frac{bh^3}{12} \\ &= \frac{5 \times 2^3}{12} \\ &= \frac{10}{3} = 3.33 \text{ m}^4. \end{aligned}$$

So in these cases I_x is greater when $h > b$ than when $h < b$.

b Yes it is true that I_x is always larger when $h > b$ than when $b > h$. To prove this, we must show that when $h > b$ we have $bh^3 > hb^3$. Remembering that both h and b are positive numbers we have

$$h > b \Rightarrow bh^3 > b^2h^2 > b^3h$$

by first multiplying both sides by the positive number bh^2 and then using again that $h > b$.

Exercise 15

a We first calculate the second moment of area:

$$\begin{aligned} I_x &= \frac{bh^3}{12} = \frac{30 \times 0.5^3}{12} \\ &= 0.3125 \text{ m}^4 \end{aligned}$$

From Exercise 12

$$M(x) = \frac{36667x^2}{2} - 1100000x + 16500000 = \frac{36667 \times 15^2}{2} - 110000 \times 15 + 16500000$$
$$= 4125037.5 \text{ Nm}$$

Now from the bending stress equation,

$$\sigma_x(y) = \frac{My}{I_x}$$
$$= \frac{4125037.5 \times 0.5}{0.3125}$$
$$= 6600060 \text{ Nm}^{-2}.$$

b The variable x is the position along the length of the wing from the fuselage – in this case, the halfway point. The variable y is the distance from the x-axis in the perpendicular direction. The variable b is the breadth of the beam and the variable h is the height of the beam.