The Cartesian Plane

In coordinate geometry, points are placed on a coordinate or *cartesian plane*. A cartesian plane has two scales that intersect at right angles to one another. The horizontal plane is always labelled the *x-axis* and the vertical is the *y-axis*. Cartesian points or *ordered pairs* are plotted on this plane in the form (x, y). Cartesian planes are used extensively in many areas, including mapping, location, aviation, economics and archeology. In fact, any time you may need to find the location of something or create a 2D graph, you could potentially use one of these planes. Plotting ordered pairs on a cartesian plane is simple. We always start at the origin (0,0), run along the x-axis the required amount of spaces and then up in the y-direction and mark our location with a point. Figure 1 shows coordinate points $A(0, 2)$ and $B(4, 6)$ with a line $\overline{AB}$ joining them. We can also say that line segment $\overline{AB} = a$.

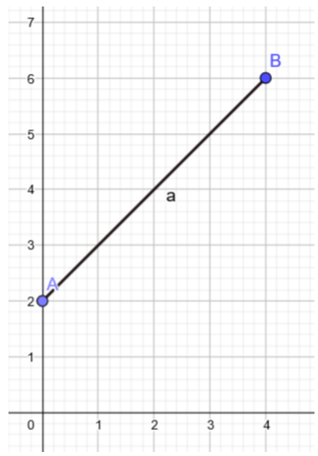

Figure 1

The Straight-Line Equation

For a line to be straight, it needs to be traced by a point and travel in a constant direction with zero curvature. In coordinate geometry, straight lines can be ***'plotted'*** on a cartesian plane using a series of connected points, or ***'sketched'*** using only two points. Each line will also have its own unique slope that we call the ***'gradient'***.

In each case, a straight-line is represented by a 'linear' equation in the form:

$$y = mx + b \qquad (1)$$

Where ***m*** represents the value of the gradient and ***b*** the value of the y – intercept or point that the line cuts the y-axis.

Activity 1 – Finding the gradient of a line

The gradient (m) of a line is another word for the slope. The higher the value for the gradient, the steeper the line is at that point and a negative gradient means that the line slopes downwards. The gradient for any line can be calculated simply using the formula:

$$m = \frac{rise}{run} \qquad (2)$$

$$= \frac{change\ in\ y}{change\ in\ x}$$

$$m = \frac{y_1+y_2}{x_1+x_2} \qquad (3)$$

Let's take another look at Figure 1.

The gradient of the line $\overline{AB}$ can be calculated using equation 3 because it already shows two coordinate points $A(0, 2)$ and $B(4, 6)$. The values x_1 and x_2 and y_1 and y_2 represent the values for x and y in each ordered pair and can be substituted directly into the formula:

$$m = \frac{y_2-y_1}{x_2-x_1}$$

$$= \frac{6-2}{4-0}$$

$$= \frac{4}{4}$$

$$= 1$$

Using the value of this gradient and the y-intercept $b = 2$, that we observe from the graph, we can now construct the linear equation for this straight line.

$$y = 1x + 2$$

$$y = x + 2$$

Activity 2 – Finding the midpoint of a line

Finding the midpoint of a line is an important aspect of successfully creating a Voronoi diagram. In a simple graph like the one in Figure 2, it is easy to see that the midpoint of the line $\overline{AB}$ will be (2,4). However, it can also be accurately represented using the **'midpoint equation'**.

$$(x, y)_{\text{MIDPOINT}} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \qquad (4)$$

For the line $\overline{AB}$ we can use this equation to calculate the coordinate of the midpoint of the line without having to rely purely on observation of the graph.

$$\overline{AB}_{\text{MIDPOINT}} = \left(\frac{0+4}{2}, \frac{2+6}{2}\right)$$

$$= (2, 4)$$

Looking at Figure 2, we can see that when plotted on the same plane, the point (2, 4) forms the midpoint of the line $\overline{AB}$ as was previously observed.

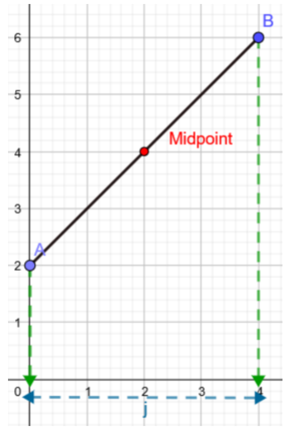

Figure 2: Graph showing the midpoint of the line AB

Activity 3 – Finding a Perpendicular Line

A perpendicular line is one that bisects another at right angles or at 90 degrees to that line. We can also tell that two straight lines are perpendicular when one of the gradients is the *negative reciprocal* of the other. The negative reciprocal can be defined using the following formula:

$$m_2 = -\frac{1}{m_1} \qquad (5)$$

Another way of saying this is:

$$m_1 . m_2 = -1 \qquad (6)$$

Let's take another look at the line $\overline{AB}$.

From Activity 1, we found that the gradient of line $\overline{AB}$ is 1. Using equation 5 or 6, we can now find that the gradient of the line perpendicular to $\overline{AB}$ is:

$$1 \times m_2 = -1$$

$$\therefore \quad m_2 = -1$$

This gives us the first piece of information needed for the equation of the line perpendicular to $\overline{AB}$. Using the straight-line equation (1):

$$y = mx + b$$

Where *m* = gradient and *b* = *y* – intercept. The equation of the perpendicular line will now have the equation:

$$y = -1x + b \qquad (7)$$

We can find the value for the *y* -intercept (*b*) by substituting any point (*x*, *y*) that falls on the perpendicular line into equation 7 and solving for *b.*

Activity 4: Finding the Perpendicular Bisector

Using the Straight-Line Equation

The perpendicular bisector is a line segment that is both perpendicular to a line and passes through its midpoint. These perpendicular bisectors are needed when constructing the Voronoi map from its Delaunay Triangulation.

For line $\overline{AB}$ we have enough information to work out what the equation of its perpendicular bisector will be. That is, we have:

- Gradient (m_2) = –1
- $\overline{AB}_{\text{MIDPOINT}} = (2, 4)$

We know that the midpoint (2, 4) falls along the perpendicular bisector, so using equation 4 and substituting the values of the midpoint, we can work out what the equation of the perpendicular line will be.

$$4 = -1 \times 2 + b$$

$$4 = -2 + b$$

$$b = 4 + 2$$

$$b = 6$$

The equation of the perpendicular bisector for line $\overline{AB}$ will therefore be:

$$y = -x + 6$$

Using the Gradient

To accurately draw a straight line however, we only need two points. If we only had the one coordinate point and the gradient, we could use this information to obtain our second point without the need for the *y* - intercept. This is often more useful when constructing the perpendicular bisectors for our Voronoi maps.

Once again, we take the information for our midpoint and gradient:

- Gradient (m_2) = –1
- $\overline{AB}_{\text{MIDPOINT}} = (2, 4)$

If we know that this perpendicular bisector will pass through the midpoint (2, 4), we can use the gradient to help work out another point that this line will pass through. To do this, we will need to take another look at Equation (2) for the gradient, $m = \frac{rise}{run}$.

Knowing that *m* for the perpendicular bisector must be –1, we now just need to find a value for the rise and the run that when substituted into equation 2 will give this value of –1. These can be *any* numbers, but to keep it simple we will choose rise = 1 and run = –1 because when substituted into the equation, they give a value of –1, the value of the gradient.

$$m = \frac{rise}{run} = -1 = \frac{+1}{-1}$$

Starting from the midpoint, we can now rise one point (move one point in the positive *y*-direction) and run negative one point (move one point in the negative *x*-direction), to plot our second coordinate point at (1, 5). Figure 3b) shows the line drawn through these two points to form the perpendicular bisector.

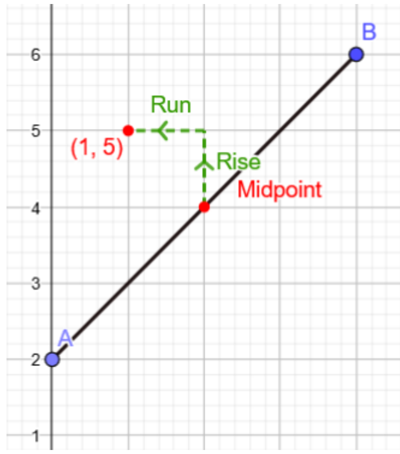

Figure 3a)

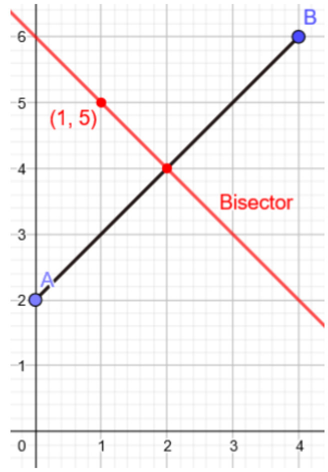

Figure 3b)

Part 2: Creating a Voronoi Diagram by hand

In Figure 4 we have added points *C* and *D* to our cartesian plane and joined each point up to create it's Delaunay Triangulation.

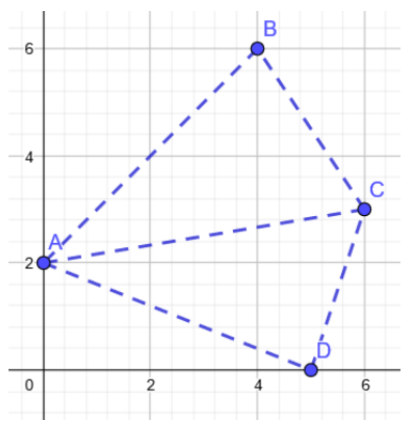

Figure 4

The Delaunay Triangulation is a set of triangles that is created by joining up points in the two-dimensional plane, so that no point lies in any triangles interior. To create our Voronoi map, we work out the perpendicular bisector for each line segment in the Delaunay Triangulation. The perpendicular bisector of line $\overline{AB}$ is indicated on the graph shown. *Exercise 1* of the activity booklet asks you to complete the information for the remaining bisectors

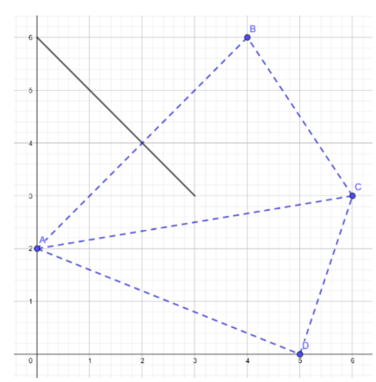

Figure 5 shows the completed midpoints and the perpendicular bisectors for each of the five segments in the Delaunay Triangulation.

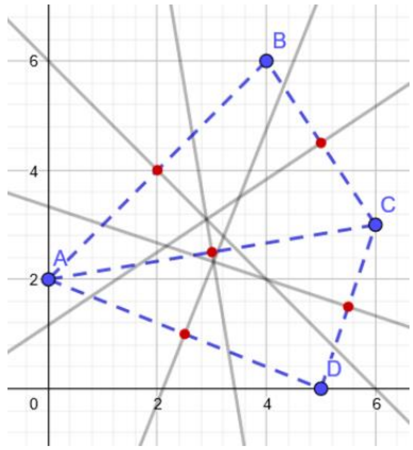

Figure 5

The Voronoi Diagram is superimposed onto these bisectors and each cell vertex corresponds to three intersecting bisectors. These steps are shown in the next three figures.

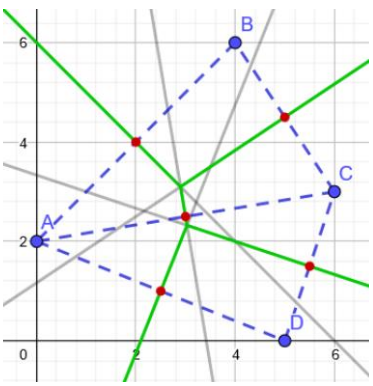

Figure 6a)

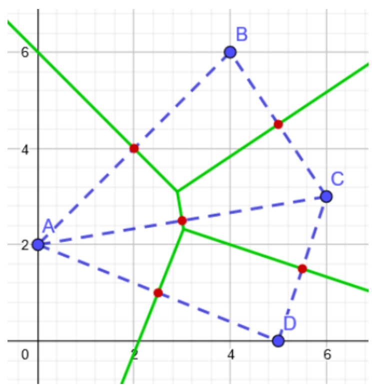

Figure 6b)

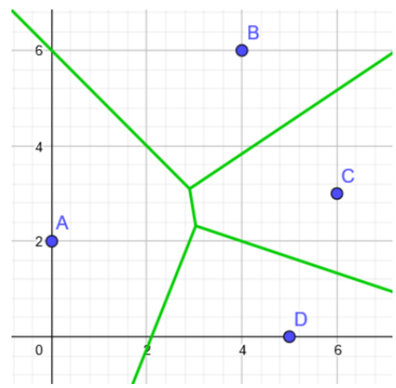

Figure 6c)

Once the midpoints, perpendicular bisectors and Delaunay Triangulation are removed as shown in Figures 6b) and 6c), our Voronoi cells become apparent and the area defined by each seed becomes clear. We can now use this diagram to determine which seed any given cartesian point is closest to within this plane.

Drawing Voronoi diagrams by hand can be a very time consuming and often complicated process. Therefore, mathematicians and scientists will often employ the use of a mathematical or imaging software to help make this much simpler. In this investigation, we will be using a mathematical platform called GeoGebra to help create our Voronoi maps and assist us with our analysis.

References

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