ALGEBRA TILES

Learning Sequence

Abstract

The following is a suggested teaching and learning sequence for using Algebra Tiles. These ideas can be used from Year 3 onwards.

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Contents

Introduction and Rationale

We use concrete materials and manipulatives in the classroom in the lower primary years but often move away from them as students get older, giving students the impression that they are 'babyish' and that it is more 'grown-up' to manage without them. This is a misguided approach and manipulatives should be used all the way through school.

Students often experience difficulty with Algebra and the notation it employs. Algebra tiles are a manipulative that can help develop student's understanding and confidence with algebra, at many different levels.

The tiles employ the area model of multiplication and this needs to be explored and understood if the students are to get the maximum understanding and benefit from their use. See the references at the end of the document for more detail on the area model.

This lesson sequence is not meant to replace a textbook but is an introduction to the use of algebra tiles. Use your usual textbook and classroom resources for exercises and practice.

Concrete, Representational, Abstract (CRA) Model

This model is used throughout the lesson sequence.

The **CRA** model is designed to move students from the:

> **Concrete** – using manipulatives, such as the Algebra Tiles; to the
>
> **Representational/Pictorial** – drawing a picture/taking a photo of the tiles; to the
>
> **Abstract** – writing the expression/equation using symbols.

Using manipulatives is an important stage in the development of student understanding. This stage is often omitted in senior school, where manipulatives are perceived as 'babyish'. Giving students the opportunity to 'play' with the tiles can help them to ***see*** and ***feel*** what is happening and can therefore, help students to develop a deeper understanding of formal algorithms.

The second stage is the representational or pictorial. In this stage students are asked to draw a diagram or take a picture of what they did. This can be just a pictorial version of the final tile arrangement or a more complex drawing explaining how they manipulated the tiles.

The abstract phase is where students move towards using numbers and symbols to show their thinking and understanding. This can be developed from the representational stage. Students can then develop (or be shown) a formal algorithm for solving the problem.

This is not a linear progression. Some students will move from the concrete to the representational to the abstract, others will jump around. Having the concrete materials available for those that want to use them is important. Students should, feel comfortable and confident using them.

There are references at the end of the paper for a deeper explanation of this model and the research behind it.

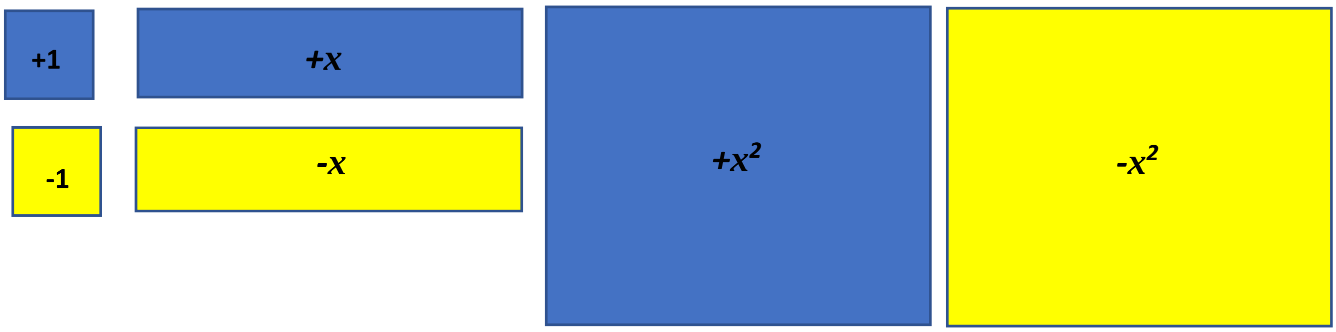
Introducing the Algebra Tiles

There are three different tiles in the Algebra Tile set, each with two colours, representing positive and negative values.

Although you can buy these tiles (see: https://www.aamt.edu.au/Webshop/Entire-catalogue/Algebra-Tiles-Australia-a-concrete-visual-area-based-model-book) they are also easy to make.

There is an interactive, online version at: http://www.drpaulswan.com.au/operating-theatre

The tiles are as follows:



The smallest tile is a 1 by 1 unit representing the constant +1 or -1

The next tile is 1 by x, representing $+x$ or $-x$.

The largest tile is x by x, representing $+x^2$ or $-x^2$.

The x dimension is intentionally **not** equivalent to a fixed number of single units, in terms of size. This is so that students do not associate the length of x with any particular value.

You could write the values on the tiles if you felt that the students needed the visual reminder.

Ideally each student should have their own set of tiles. To make a set of tiles:

1. You will need two colours of cardboard, or thin foam (preferably one of these should be with an adhesive side). We suggest that you use a ‘hot’ and a ‘cool’ colour, such as blue/green or red/yellow. (Choose colours that are easy for students who are colour-blind to recognise.)
2. If you are using cardboard:
   a. Photocopy the template below on to one piece and then stick them together and laminate
   b. Cut out the tiles using the template
3. If you are using foam:
   a. Stick the two colours together
   b. Using the following dimensions draw and then cut out the tiles:
      i. 1-unit tiles – 1.5cm x 1.5cm
      ii. x tiles – 1.5cm x 7 cm
      iii. x^2 tiles – 7cm x 7cm

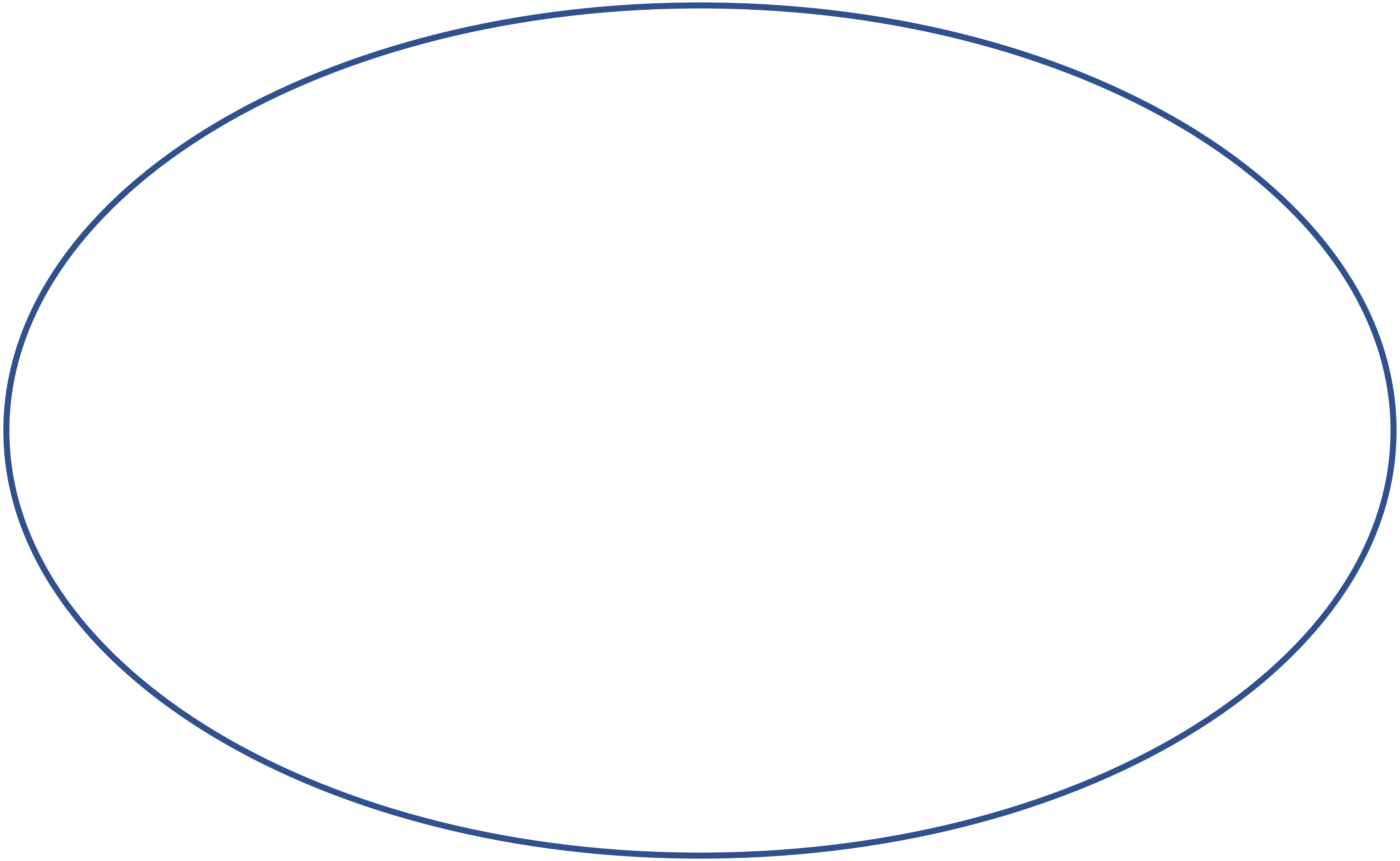
Creating the tiles with the students is a good lesson in itself – especially if you measure out and cut them. It is also a good introduction into the concept of the area model.

Templates

a. the template for making Algebra Tiles
b. the multiplication space
c. the general workspace

Students will need an A3, double-sided, laminated sheet with the two workspaces as shown in the templates below. The first is for the modelling of multiplication and the second is an open workspace. Having these laminated means that the students can annotate what they are doing, with whiteboard markers, as they are working, rather than taking notes separately.

X

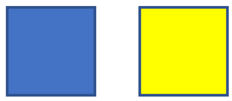


Zero-Sum Pair

This concept is key to the understanding of algebra tiles.

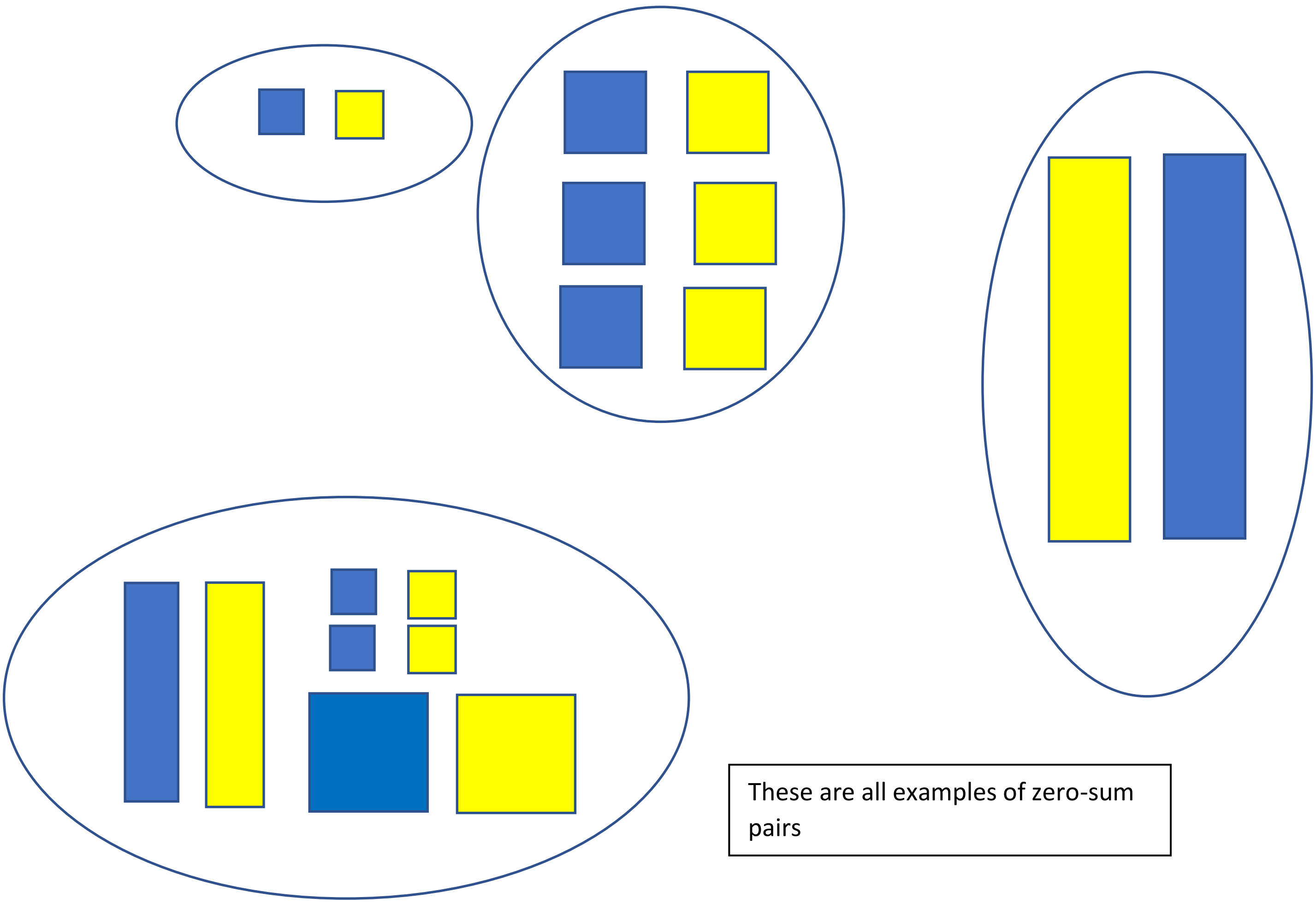
The two colours represent positive and negative values.

Show the students two tiles, the same size but different colours:



Ask the students what they see that is the same and what is different.

Inform the students that one represents a positive value and one a negative value of the same magnitude. When put together they make zero, just as -1 + 1 = 0. This is called the zero-principle and the two tiles form a **zero-sum pair**. This is an important concept and the students will need to be familiar with it as they work with the tiles.



Modelling Integers

The first stage to using algebra tiles is to model different integers in the open workspace.

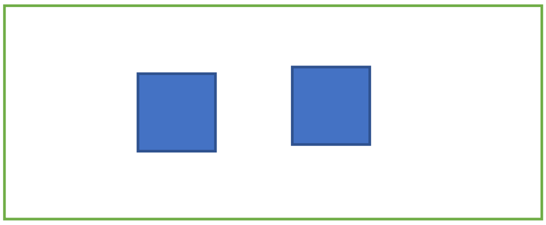
Allow the students ample time to familiarise themselves with the tiles, the meanings of each colour and the concept of zero-sum pairs.

Ask then to model numbers in multiple ways – using the zero-sum pairs.

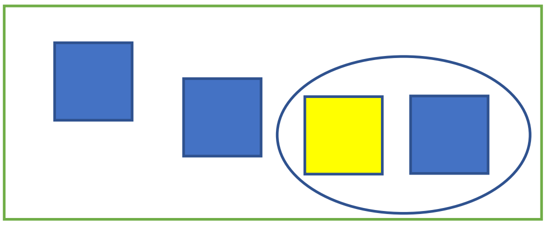
They should use the tiles, draw representations of what they have done and then write their ideas using symbols.

e.g.

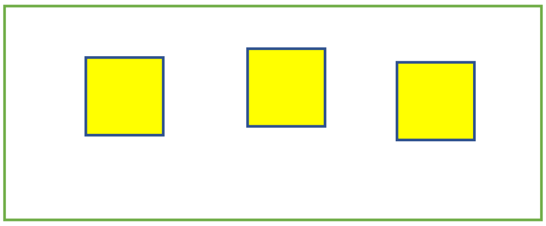
+2



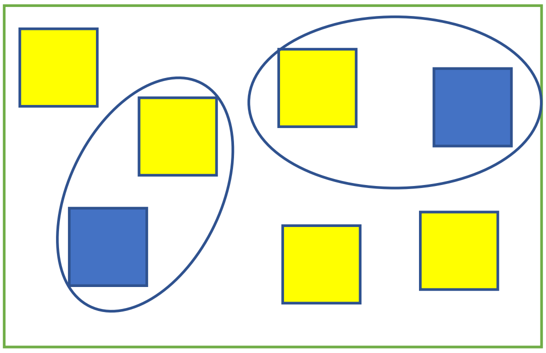
(+2) + (-1 + 1) = +2



-3



(-3) + (-1 + 1) + (-1 + 1) = -3



I have used the plus-sign to indicate a positive integer in these examples. It is important that students become aware that a number with no sign is the 'lazy mathematicians' way of writing a positive number! As we assume that most numbers are positive we do not indicate it.

Here, as I want to clearly distinguish between positive and negative quantities, I have used the '+' symbol. This can be dropped as the students become more familiar and confident with manipulating the tiles.

Addition and subtraction of integers

The addition of integers is the first operation to model with students.

It is important to distinguish between the direction of a number and the operation being performed.

For example:

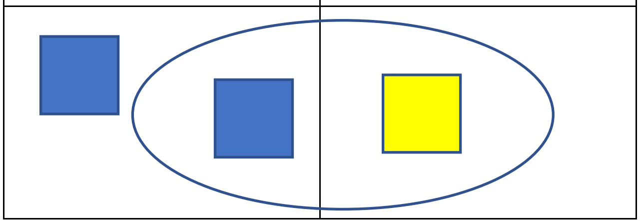
(+2) + (-1) = +1 Should be read as: 2 add, negative 1

(+2) - (-1) = +3 Should be read as: 2 subtract, negative 1

The operation ‘subtract’ can be modelled as the opposite of ‘add’. In terms of Algebra Tiles this means ‘turn over’.

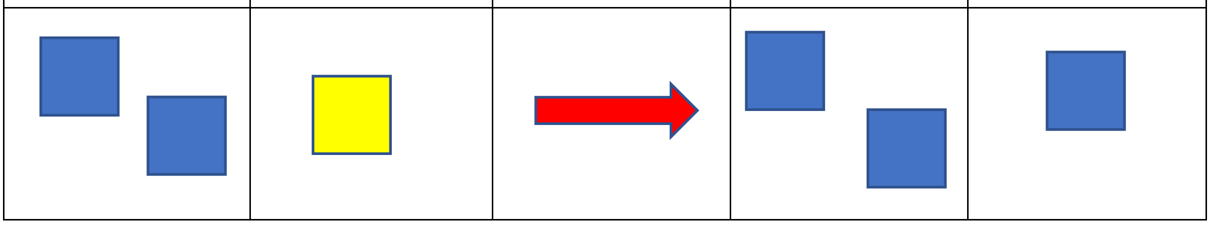
For the examples above:

(+2) + (-1) = +1	
+2	**-1**



The circled pair of tiles form a zero-sum pair

(+2) - (-1) = +3				
+2	**-1**		**+2**	**- (-1)**

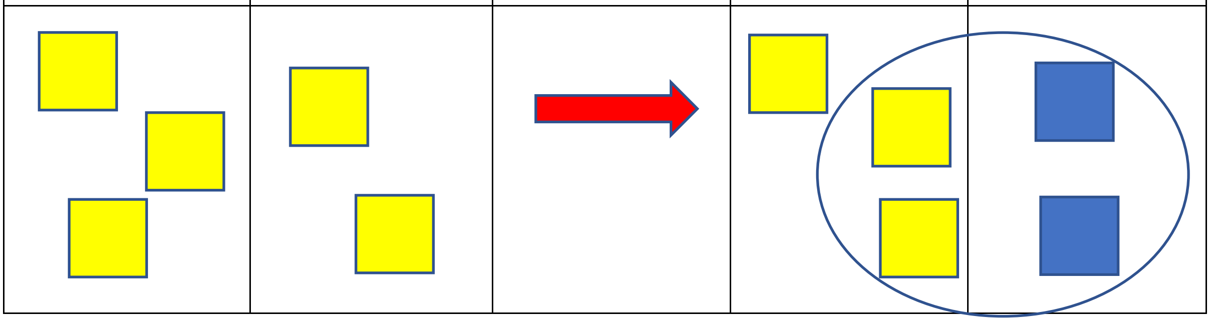


Model the numbers

Turn the second number over to model the operation ‘subtract’

Another example:

(-3) - (-2) = -1				
-3	**-2**		**-3**	**- (-2)**



Model the two numbers in the equation

The second number is turned-over as the operation is 'subtract'.

Then the zero-sum pairs are identified and removed, to leave the result (answer).

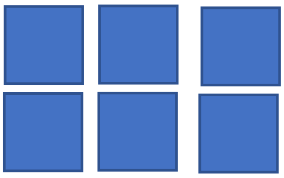
Give students ample time to practice simple addition and subtraction problems to model and solve.

Area model of multiplication

The area model of multiplication uses the same process as arrays. Area is found by multiplying two values. Therefore, by arranging the algebra tiles in a rectangle we can find the product.

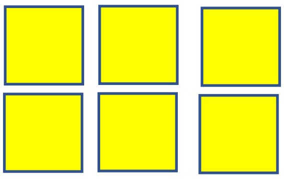
To find the product of two numbers arrange the tiles in a rectangle with the given dimensions.

e.g.

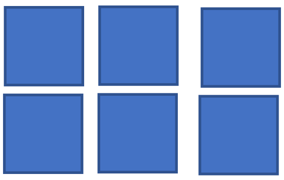
	2 X 3 = 6
X	**3**
2	

When multiplying by a negative number arrange as if positive then turn-over the tiles for each negative number.

If one of the multipliers is negative turn-over the tiles once – they will all be negative

	2 X -3 = -6
X	**-3**
2	

If both multipliers are negative turn-over twice – they will return to positive

	-2 X -3 = 6
X	**-3**
-2	

In general, where the signs of the multipliers are different the result is negative, where they are the same the result is positive. This result should become apparent to students as they use the tiles to model multiplication.

The area model is useful when looking at large number multiplication and partitioning numbers.

e.g. **27 x 43**

x	**20**	**7**
40	40 x 20 = 800	40 x 7 =280
3	3 x 20 = 60	3 x 7 = 21

27 x 43 = 800 + 280 + 60 + 21 = 1161

This model will lead into the multiplication of linear and quadratic expressions too. Students should be able to relate the different areas to the different parts of the expressions being multiplied. It should help them to see what is meant by ‘multiplying everything by everything’ or the FOIL method.

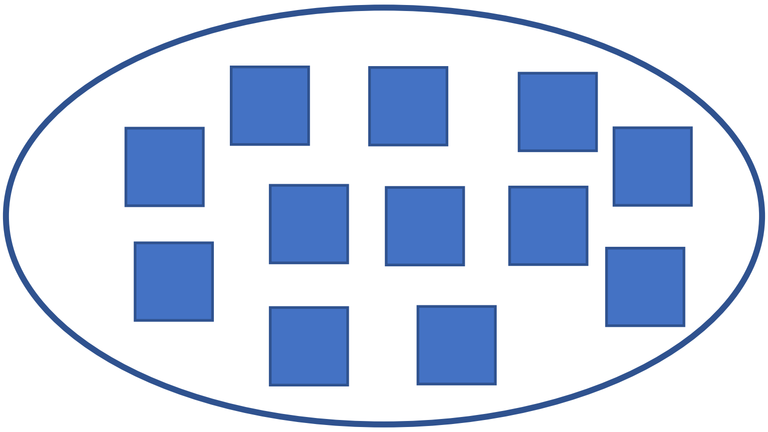
There is a reference at the end to a James Tanton video explain the area model.

Using the area model for division

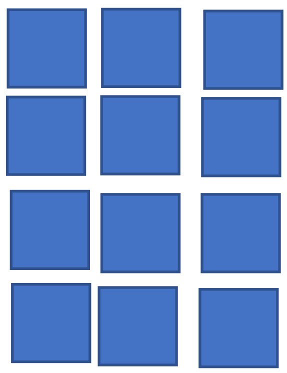
The area model can also be used for division.

Here, you take the given number of tiles (the dividend) and arrange them in to a rectangle, with one side set to the given dimension (the divisor). The other dimension will then be answer to the problem (the quotient).

e.g. for **12 / 4**



Take the 12 tiles and arrange in a rectangle with 4 rows. The answer will then be the resulting number of columns.

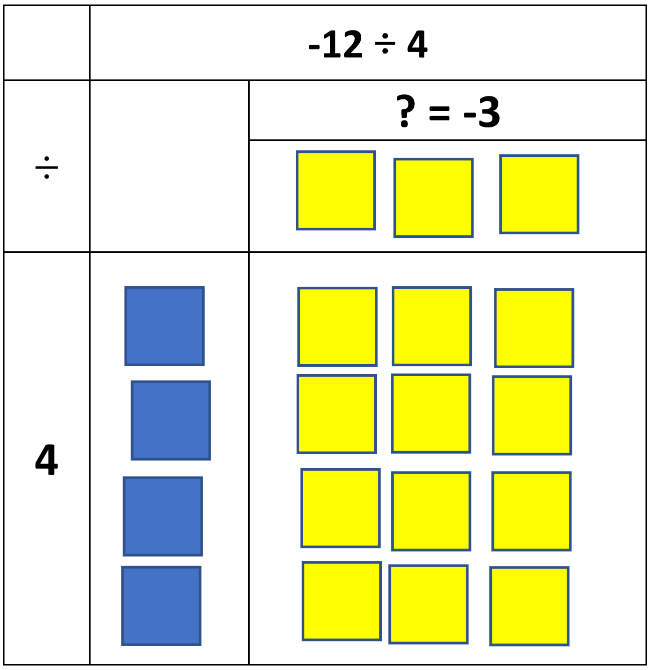
	12 ÷ 4
÷	**? = 3**
4	

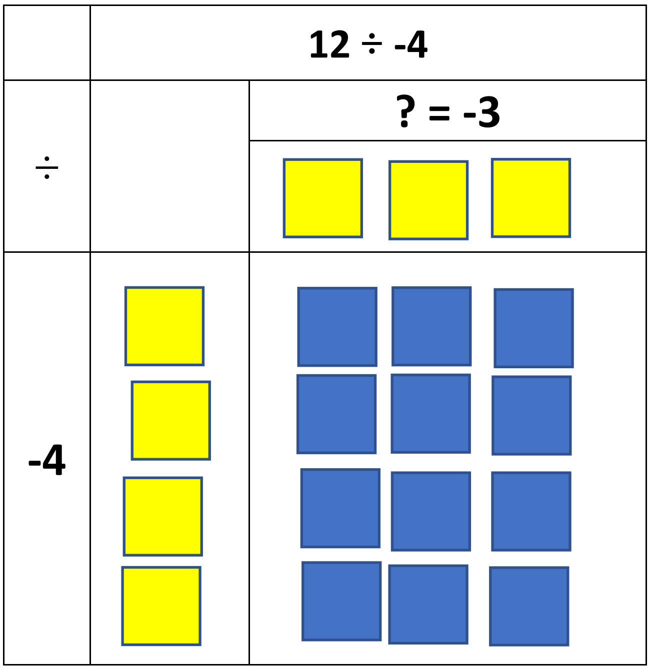
If the divisor is negative, then **either** the dividend or quotient must be negative.

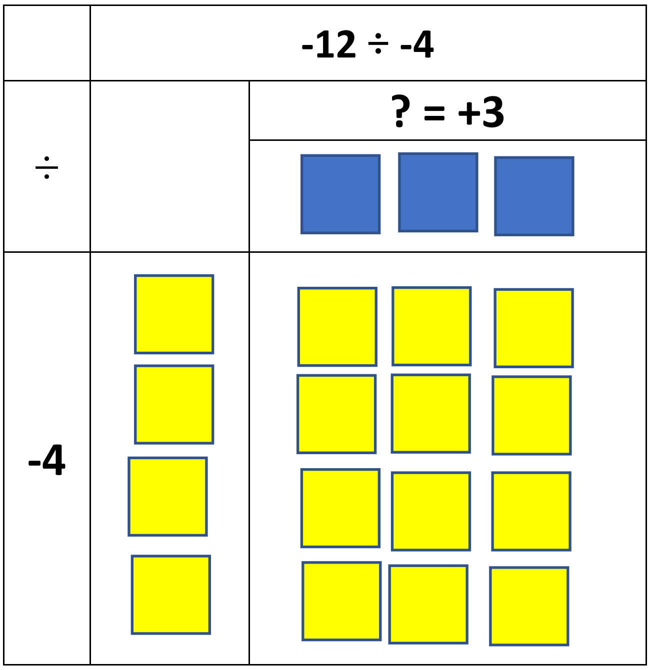
If divisor is positive and the dividend is negative, then the quotient **must** be negative.

These results should become obvious as the students explore division using the tiles.

For example:



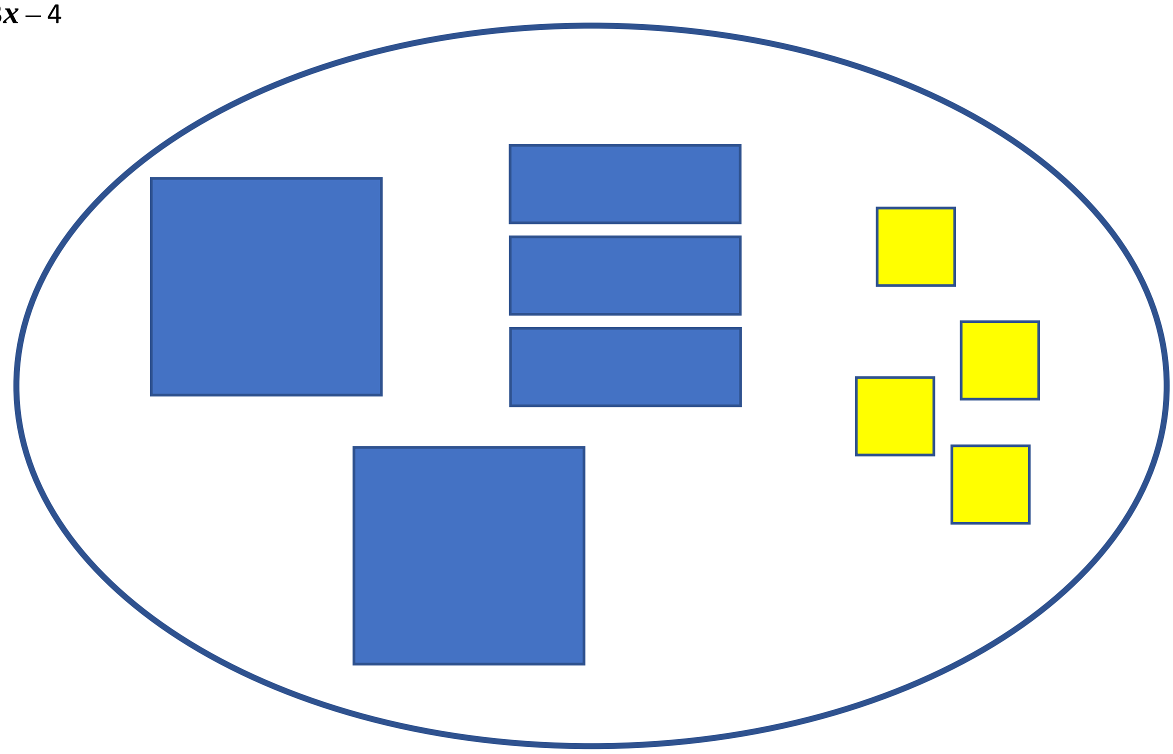




Forming algebraic expressions

We form algebraic expressions by taking the required number of algebra tiles and turning them to show the appropriate colour.

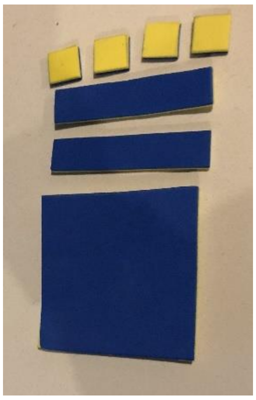
e.g. $2x^2 + 3x - 4$



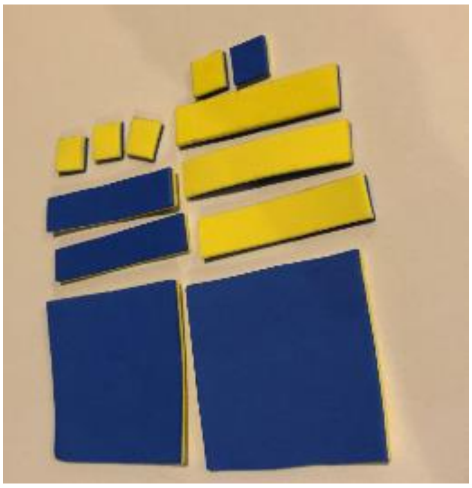
Adding and subtraction algebraic expressions

To add expressions first form the expressions then put them together. Remove any zero-sum pairs that are formed and count the remaining tiles.

e.g. $(x^2 + 2x - 4) \quad + \quad (x^2 - 3x + 1)$

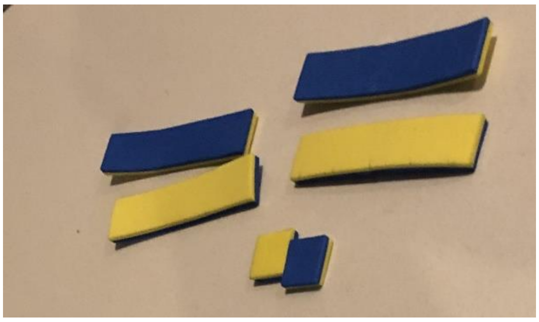


Form each expression

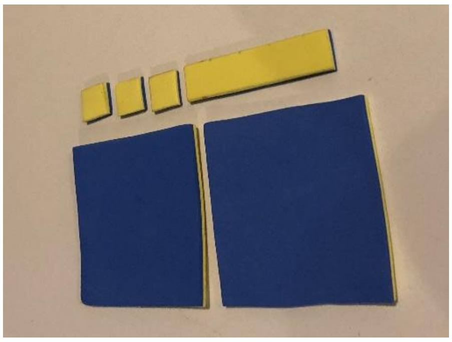


Put the two sets of tiles together, arranging the same sized tiles together (collecting like terms)

Remove any zero-sum pairs formed



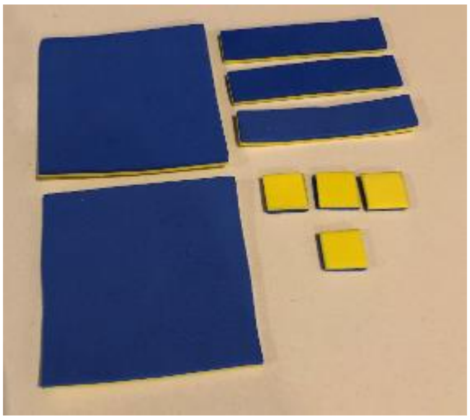
The remaining tiles form the resulting expression

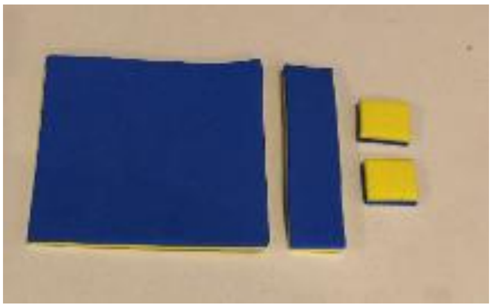


$$\mathbf{2x^2 - x - 3}$$

To subtract one expression from another, first form the two expressions:

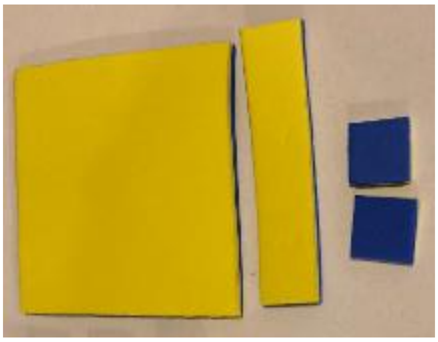
$(2x^2 + 3x - 4) \quad - \quad (x^2 + x - 2)$



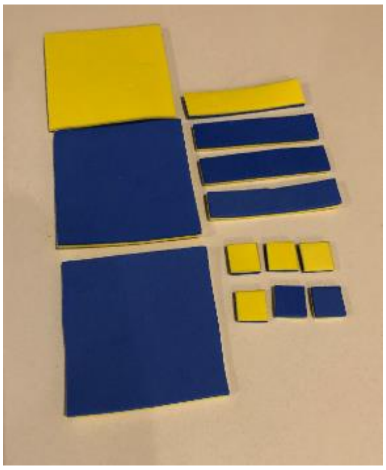


Before you combine the expressions turn all the elements of the second expression over.

The reason for turning the second set of elements over is that subtraction is the opposite of addition. You are in effect multiplying all the elements by -1.

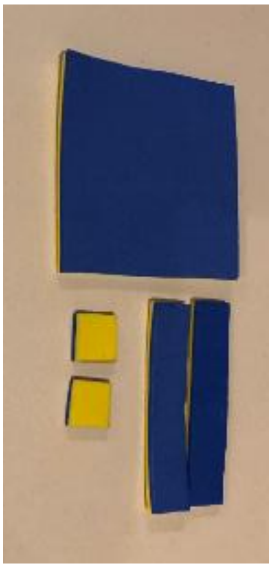


Then combine



and remove any zero-sum pairs

to find the solution



$\boldsymbol{x^2 + 2x - 2}$

Substitution

We can use the algebra tiles to illustrate the meaning of ***substitution***, in algebra.

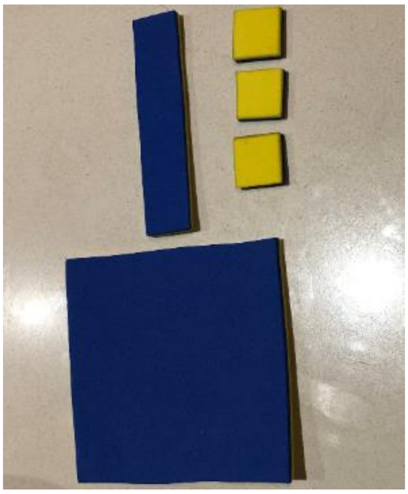
Each x-tile can be replaced by the given number of unit tiles.

To replace any x^2 tiles, form a square with side-length equal to the given value.

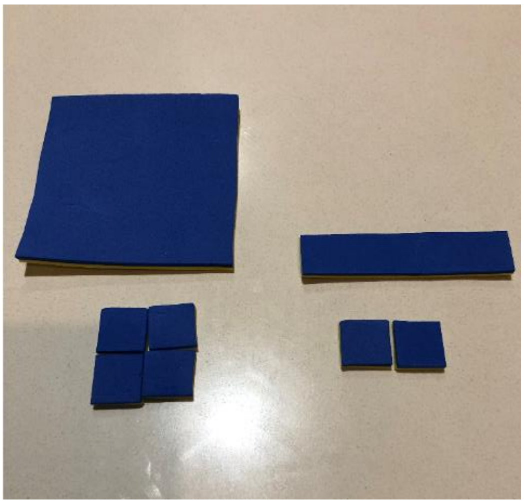
After replacing the x^2 and x-tiles, remove any zero-sum pairs to find the total.

e.g. find the value of $x^2 + x - 3$, when $x = 2$

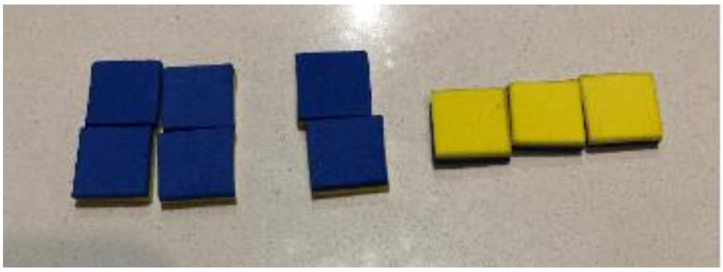
The original expression



Replace the x^2 with 4 unit-tiles and the x with two unit-tiles



Giving:



Remove the zero-sum pairs:

The value of $x^2 + x - 3$, when $x = 2$

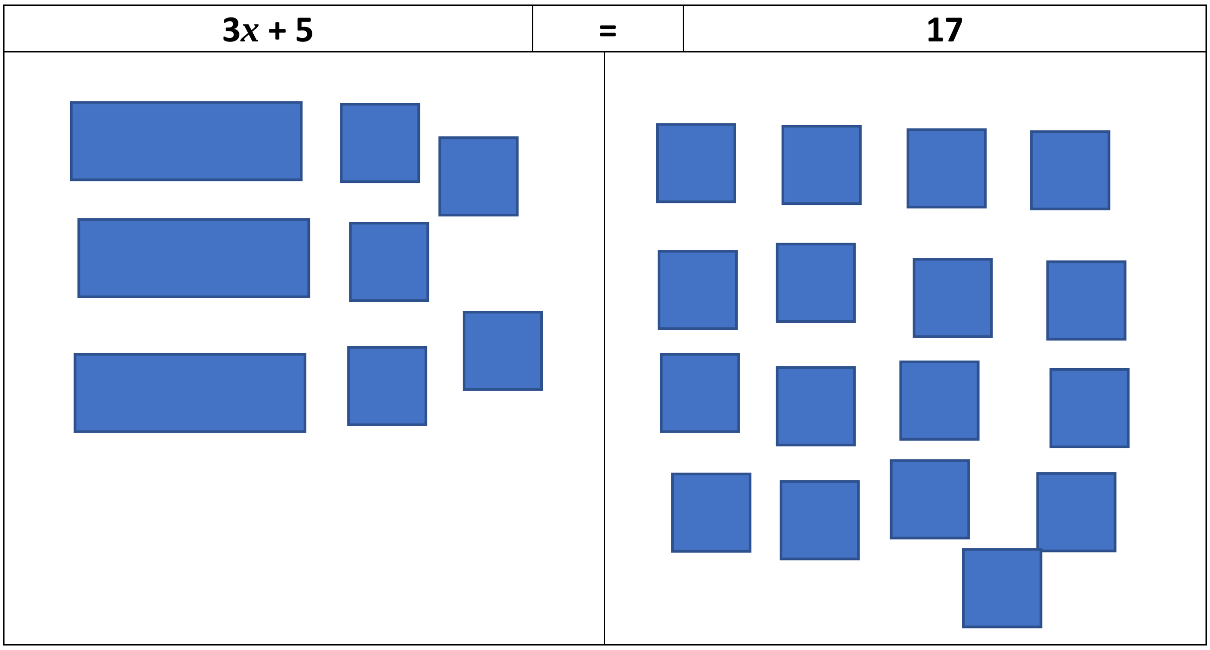
$(2)^2 + (2) - 3 = 4 + 2 - 3 = 3$

Solving linear equations

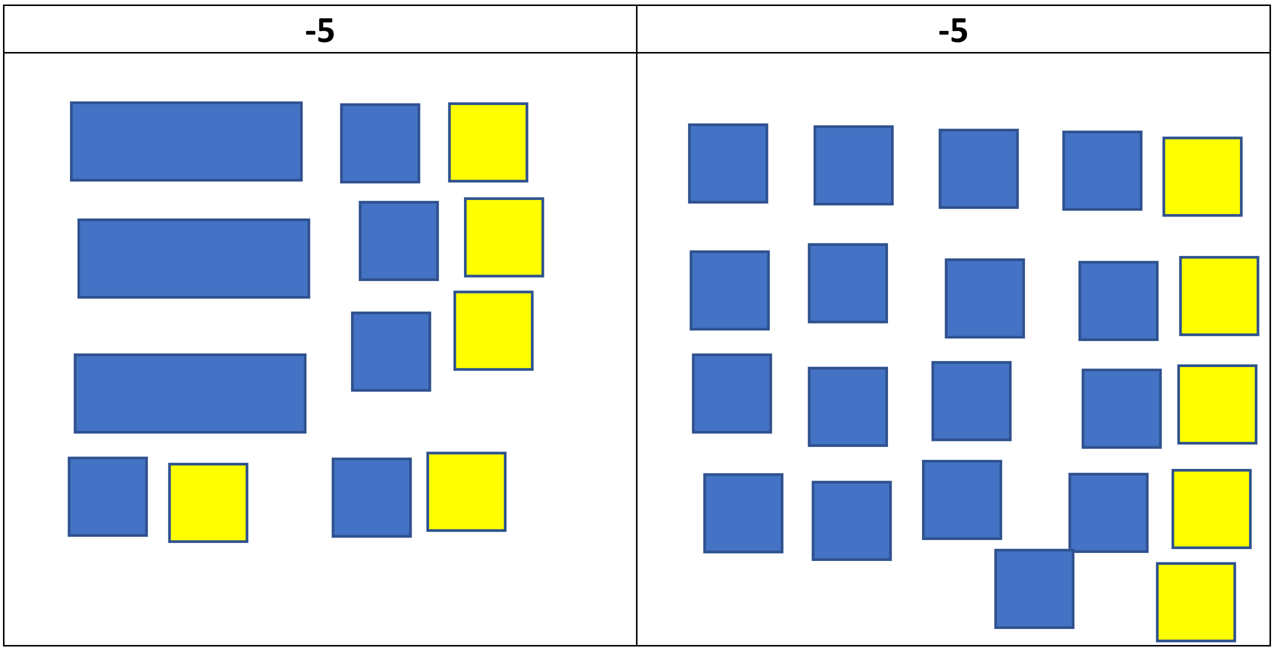
We can use algebra tiles to solve linear equations. This is a good way to illustrate the balancing method and for students to get a ‘feel’ for ‘doing the same to both sides of an equation’ and why we do this.

e.g. Solve **3x + 5 = 17**

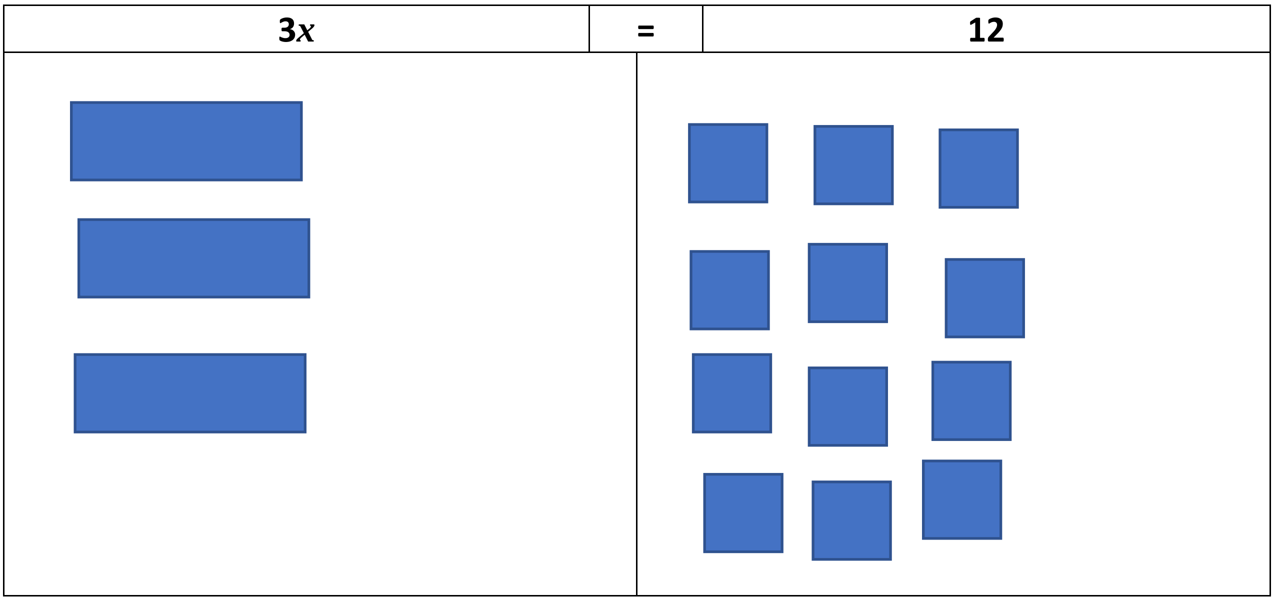
First model the problem:



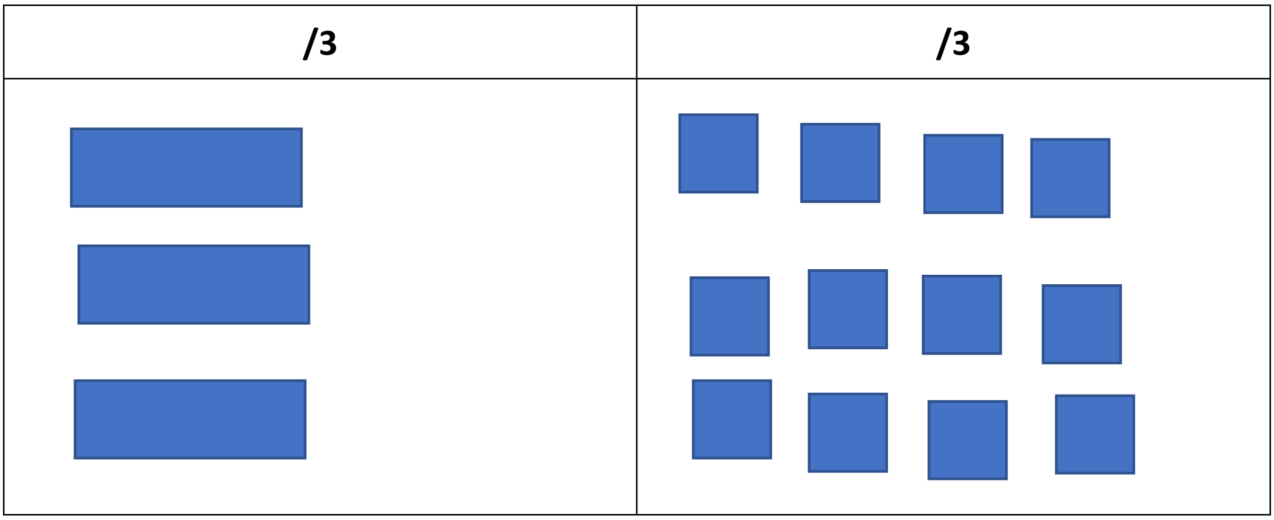
Add tiles to form zero-sum pairs:



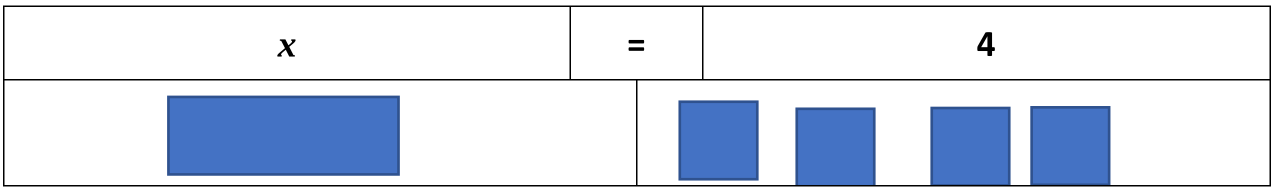
Remove the zero-sum pairs:



Now group the unit-tiles by the number of x-tiles:



Identify the value of one x-tile



Each x-tile is equivalent to 4 unit-tiles.

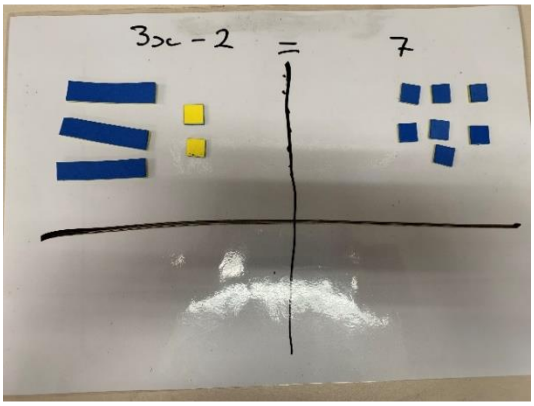
Giving a visual demonstration using concrete materials like algebra tiles can help students develop a stronger understanding of the formal algorithm.

It is important that students have an appreciation for what ‘solving’ means in this context. Remind students that we are trying to discover the numerical value of the unknown quantity x.

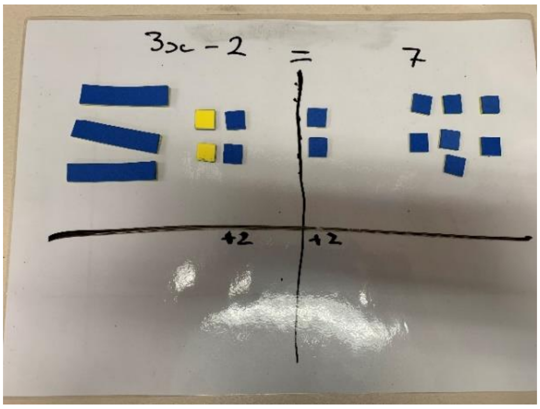
Note: algebra tiles can only be used if the solution is an integer value. It is important to give students other examples where this is not the case so that students do not, mistakenly, believe that solutions will always be integers. These can be introduced once students are comfortable working in the abstract mode.

The following are some more examples of solving equations with algebra tiles.

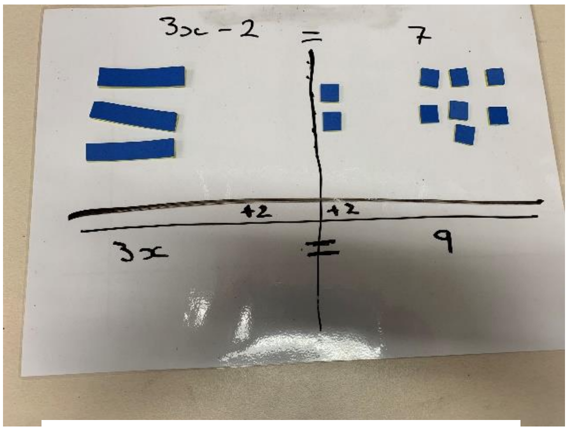
Example 1: $\mathbf{3x - 2 = 7}$



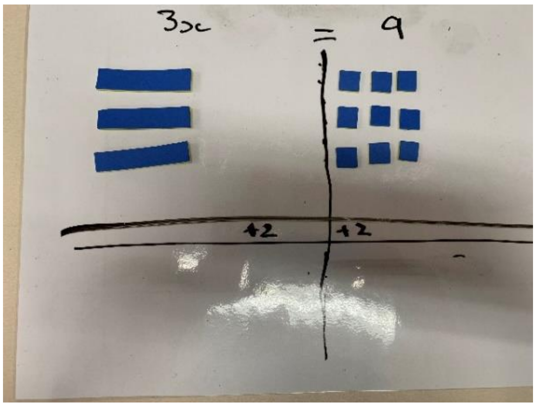
Model the question

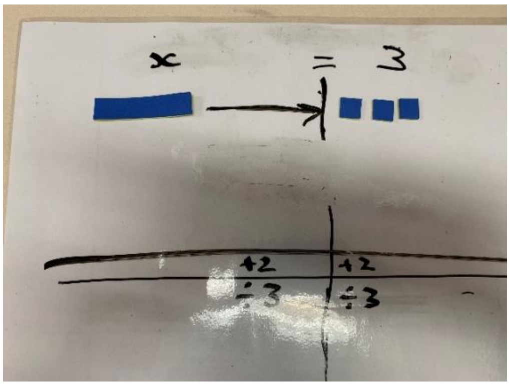


As we are trying to find the value of x we need to remove elements until we have an x-tile on its own. Here, we first remove the unit-tiles on the same side as the x-tiles. In this case by adding two unit-tiles to each side.



This is the result of adding the tiles.

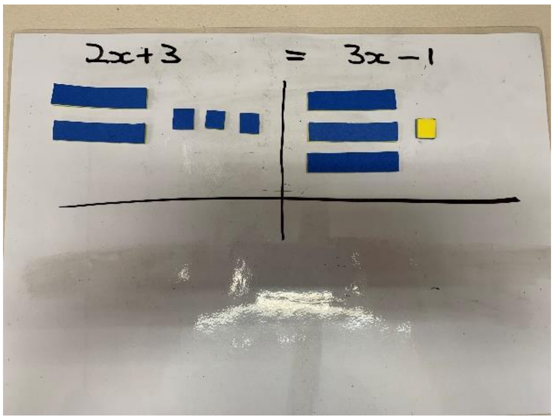




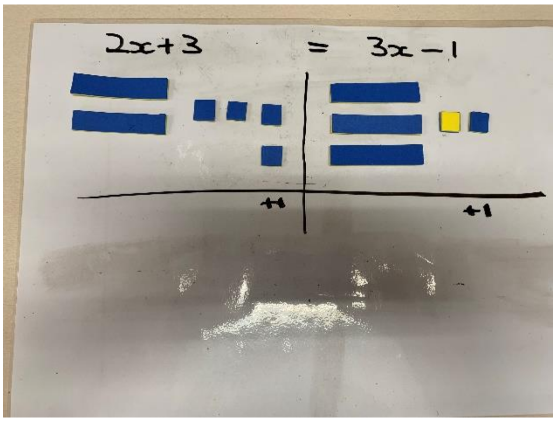
To find the value of one x-tile we need to group the unit-tiles against the x-tiles.

This shows us that one x -tile is equivalent to 3 unit-tiles.

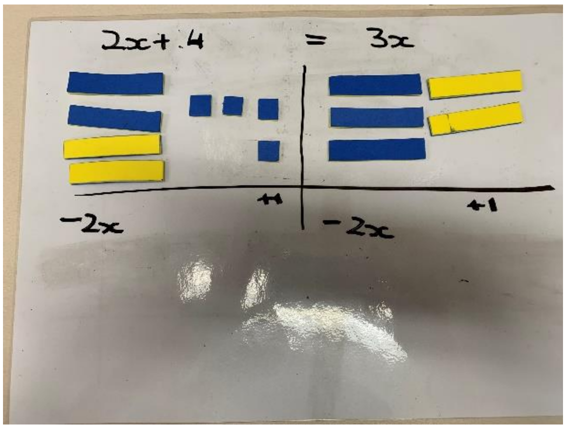
Example 2: **2x + 3 = 3x - 1**



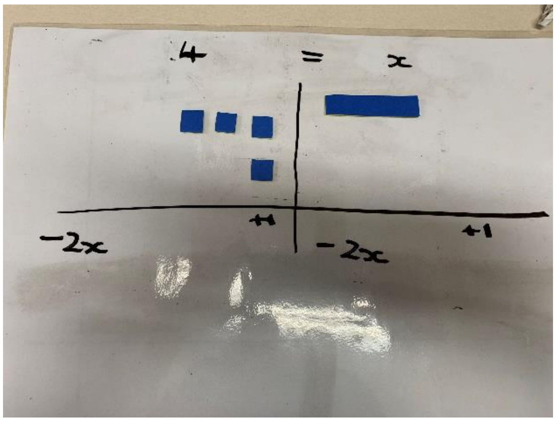
Model the question



Remove negative constants by adding positive unit-tiles to form zero-sum pairs



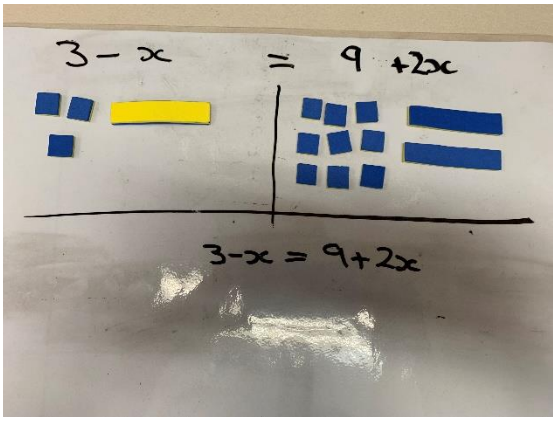
As we only want x-values on one side we need to remove the smaller x-value by adding negative x-tiles to form zero-sum pairs



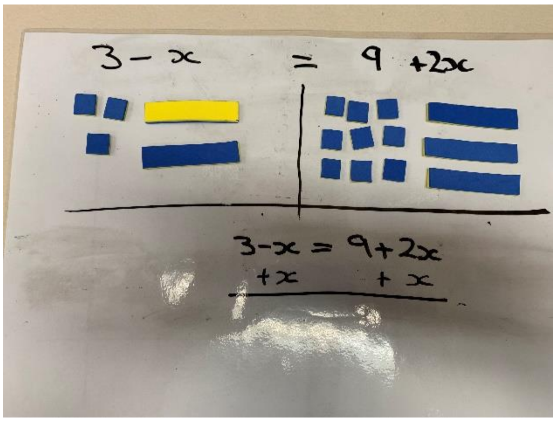
Removing the zero-sum pairs gives us the value of the x-tile. In this case x = 4.

It is important to show that the x does not have to be on the left-hand side.

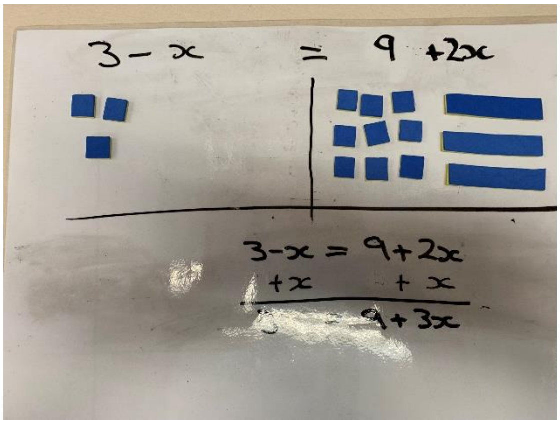
Example 3: **3 – x = 9 + 2x**



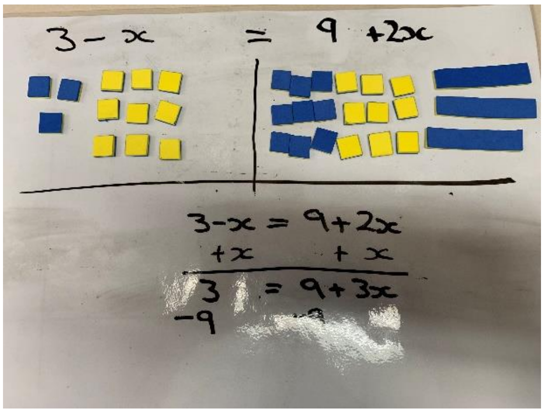
Model the question



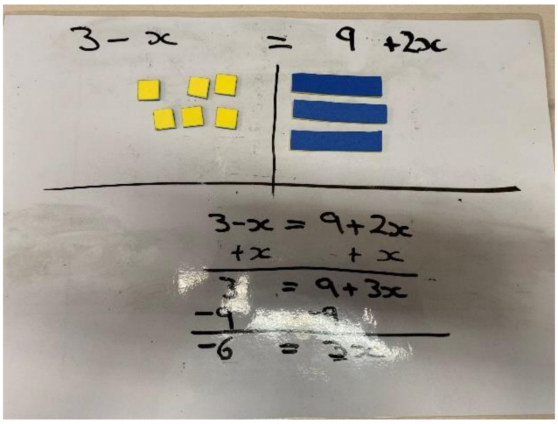
Always deal with negative values first – in this case by adding an x-tile to form a zero-sum pair



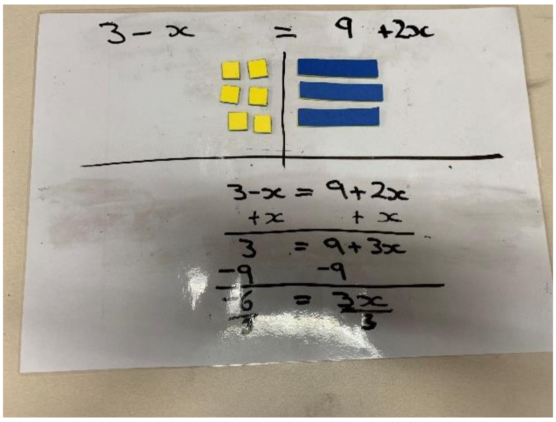
Remove the zero-sum pair



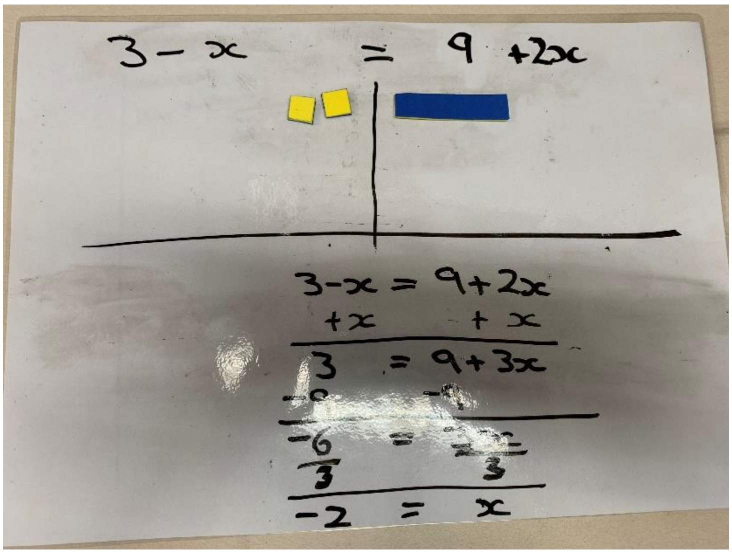
Add negative unit-tiles to form zero-sum pairs to isolate the x-tiles



Remove the zero-sum pairs



Group the unit-tiles to correspond with the x-tiles



Display the value corresponding to one x-tile.

In this case x = -2

Students should be aware that x is not always a positive value.

In the above examples I have tried to show different styles of equations and different ways of annotating.

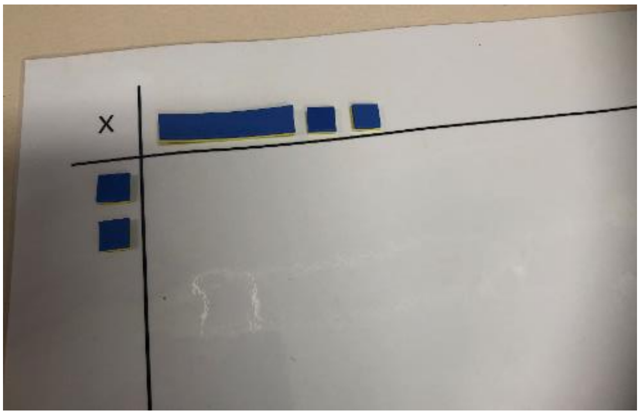
Allow the students to discover their own ways of annotating, no one style is correct. It is more important that students find a way of solving these equations themselves and develop an abstract representation that works for them. You may wish to show them some traditional approaches/algorithms, but I would suggest you do this only when the students are comfortable and confident in solving simple equations.

Multiplying algebraic expressions by integers

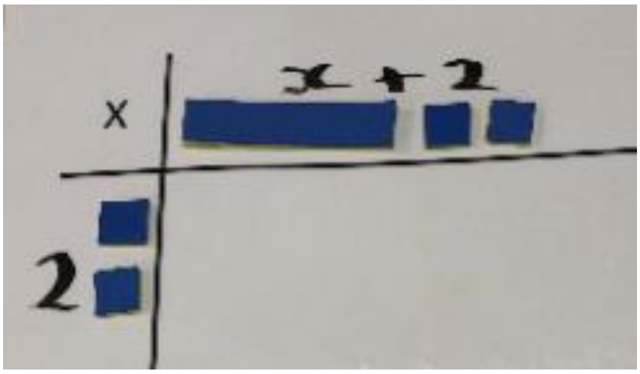
The first stage in multiplying algebraic expressions is to multiply them by an integer.

To do this using the algebra tiles use the multiplying template to arrange the question and then fill-in the rectangle. It is a good idea to suggest to the students to arrange the tiles so that the x -tiles are first and then the unit-tiles. This reflects the arrangement in the area model of multiplication.

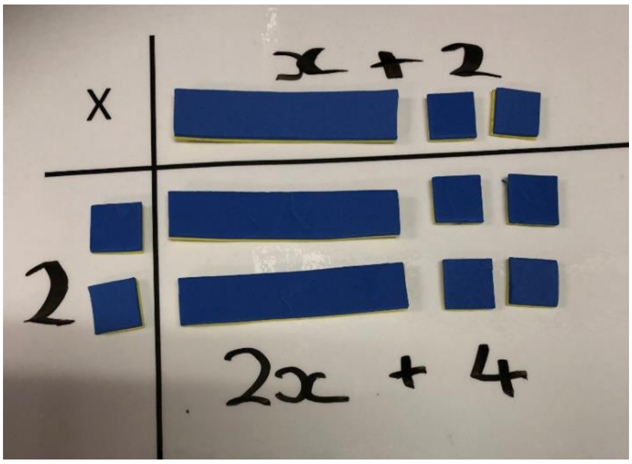
e.g. **2 x (x + 2)**



Start by putting the multiplicands in the template.



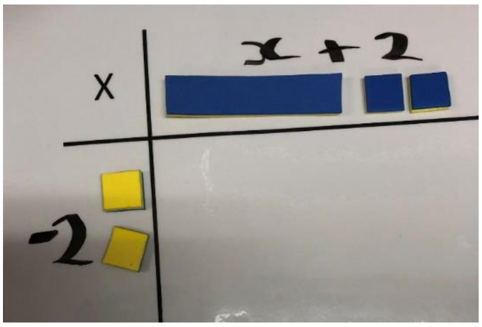
As the template is laminated the students can annotate what they are doing on the work-surface.

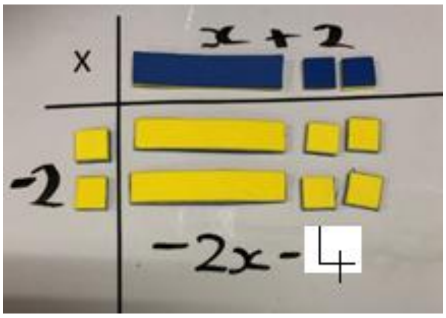


Fill in the rectangle and count the tiles to solve the problem.

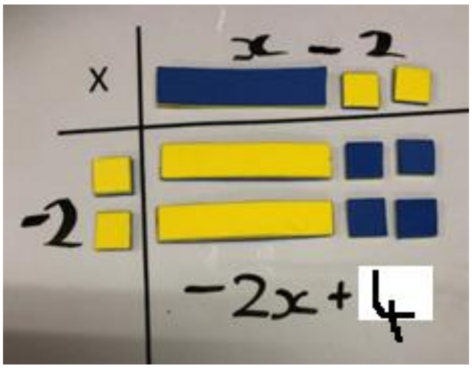
You can change the problem to multiply by -2 by turning the tiles over.

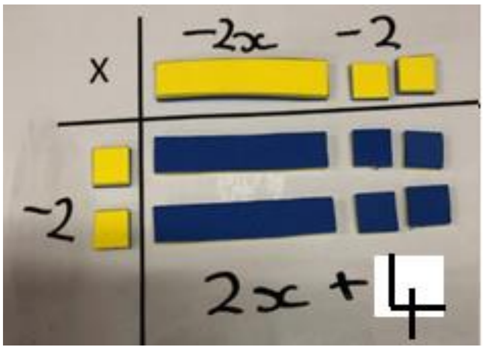
e.g.





Or change the question to see how this will affect the result.

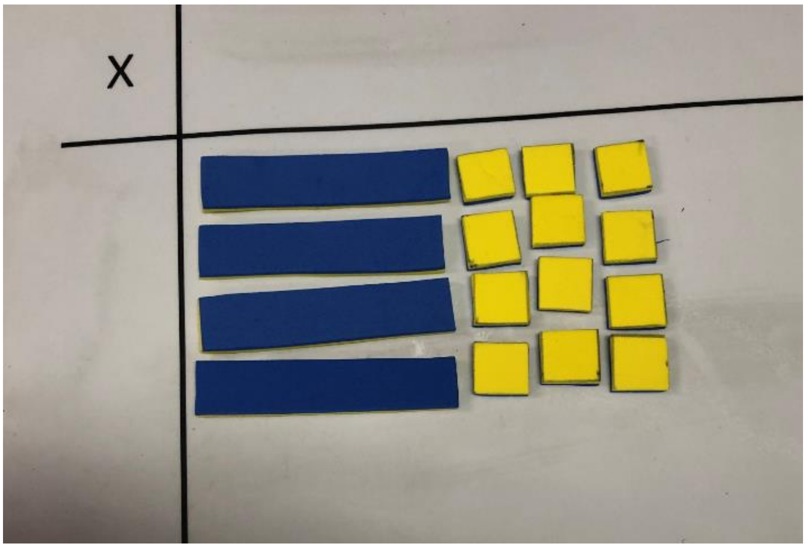




But this is the same as 2 x (x + 2)!?!

Or show a product and ask what the multiplicands could have been?

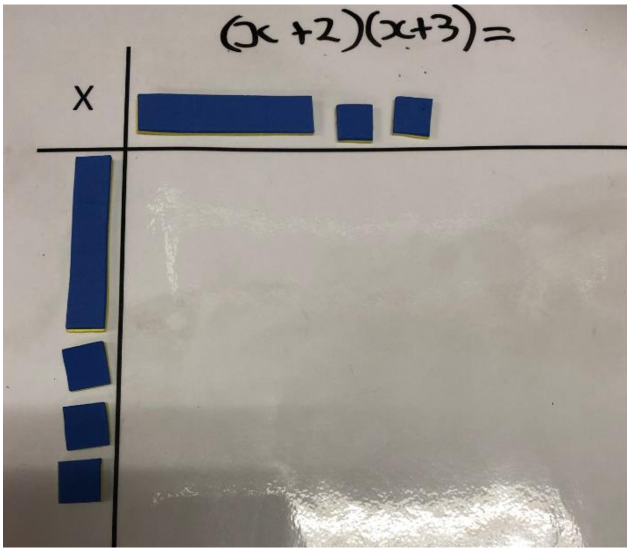
Can you do this another way?

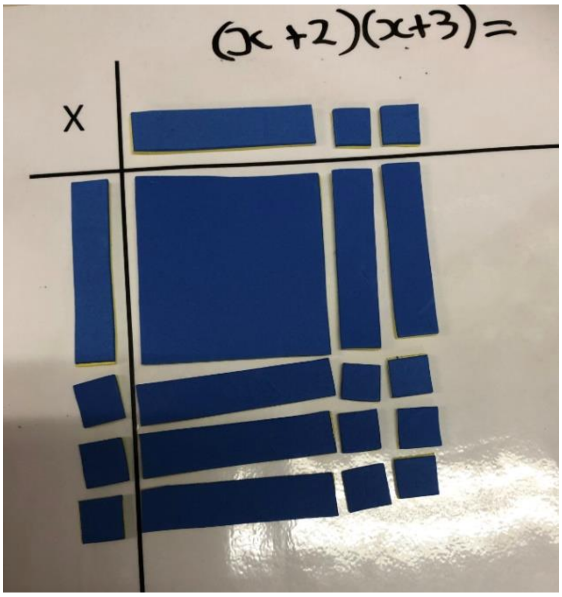


Multiplying two linear (binomial) expressions to form a quadratic

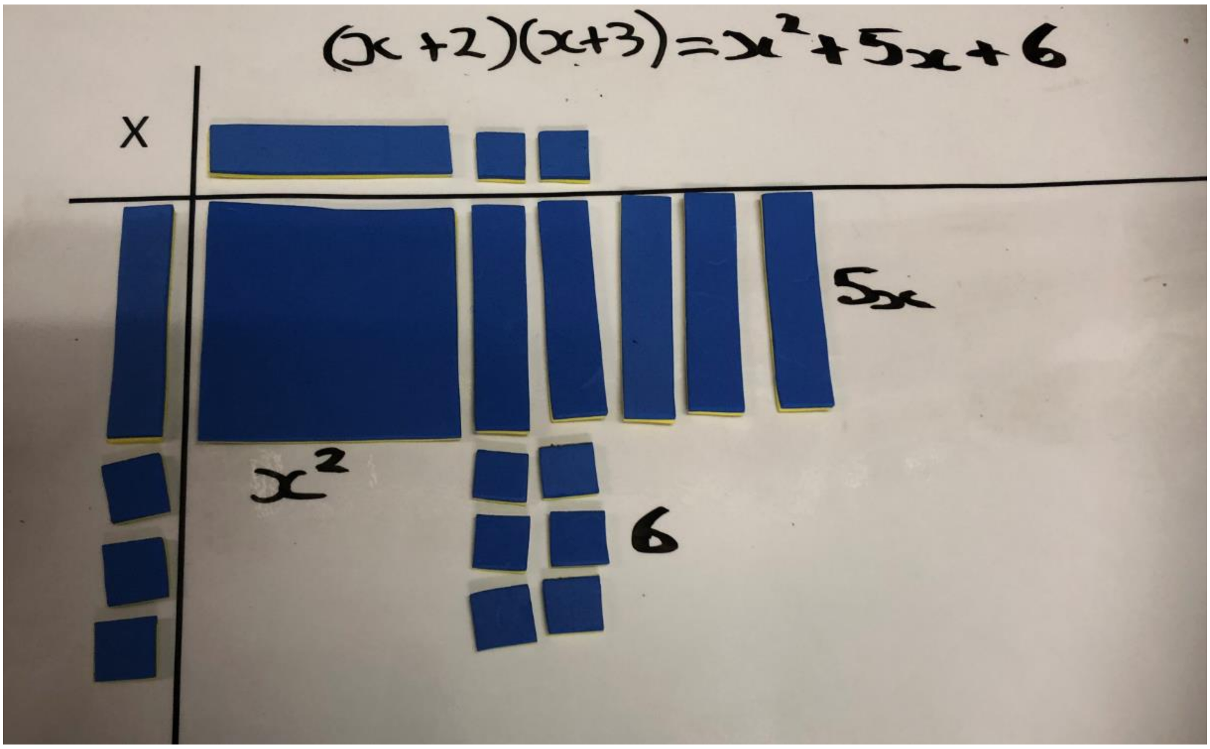
You do this the same way as above but when you multiply x by x you fill in the space with the x^2 tile.

e.g. **$(x + 2)\ (x + 3)$**



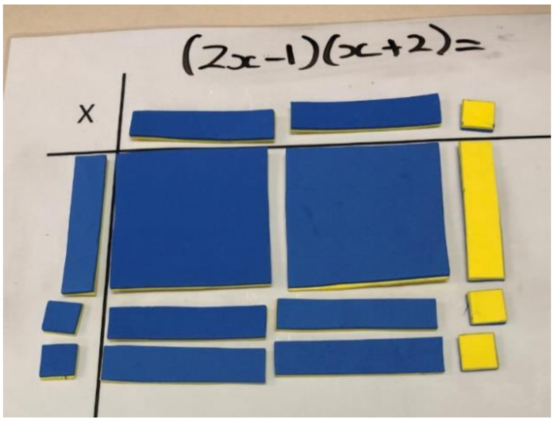


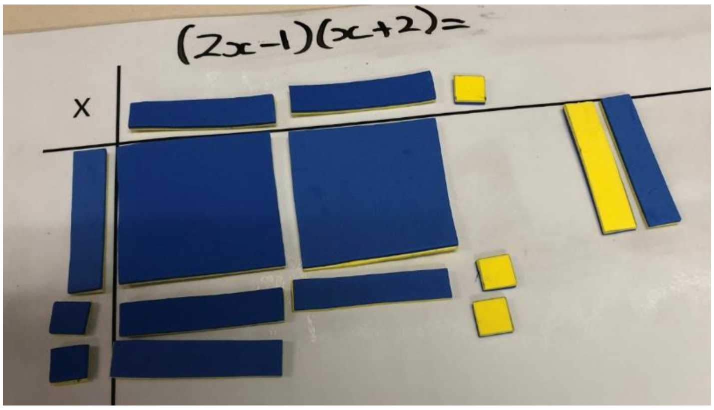
You can find the result by counting the individual tiles:



It is important that students get time to practice this and use the CRA model. Some will move away from the tiles to drawing pictures and then to symbolic notion earlier than others.

Practice at multiplying directed numbers is important too, as this is a skill that students often struggle with.





Remember to remove zero-sum pairs

Students may start to notice patterns and should be allowed to explore these; this may lead them to factorisation.

Students should be encouraged to relate the areas of the rectangle formed by the algebra tiles, to the area model for multiplication.

X	**2x**	**-1**
x	$2x^2$	$-x$
+2	$+4x$	-2

$$(2x - 1)(x + 2) = 2x^2 - x + 4x - 2 = 2x^2 + 3x - 2$$

This is a good way to move to the **representation** element of the CRA model. **Representation** does not only mean that students draw a picture of the tiles. Drawing the area model, as above, is another way of representing the problem.

Factorising by an integer factor

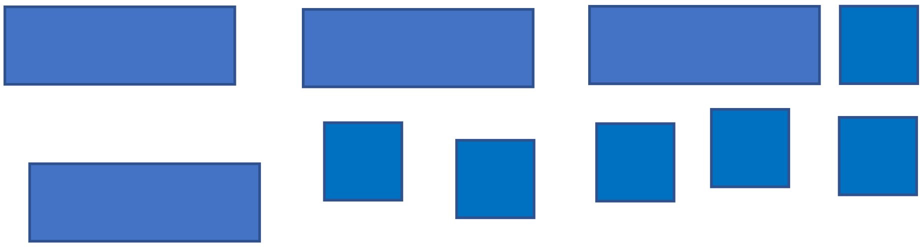
Once students are familiar and comfortable with multiplying linear expressions you can move on to factorising.

Start with factorising by an integer factor only.

Use the same template as for multiplication.

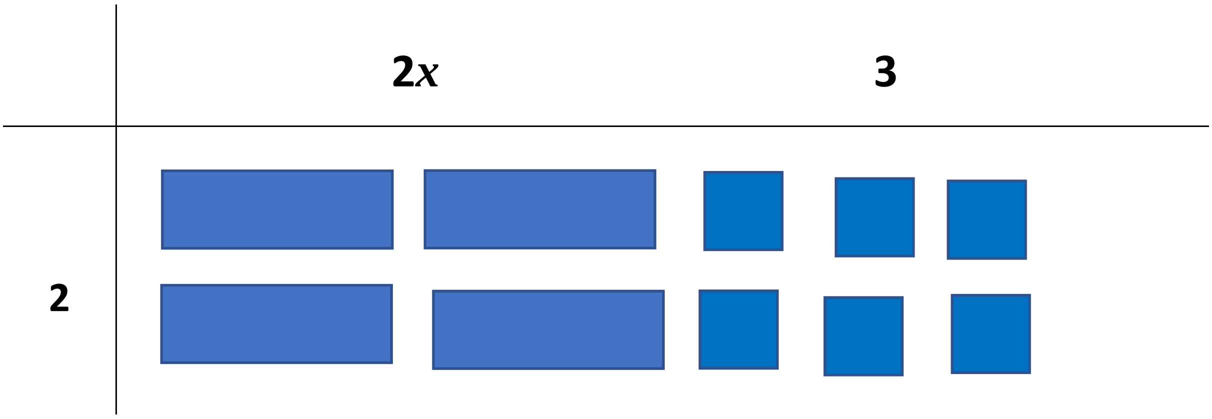
The students must arrange the tiles into a rectangle and then identify the factors.

e.g. If they were given the following tiles:



Here we have 4 x-tiles and 6 unit-tiles. It is important for students to identify the tiles they have before they start to factorise. They will need to have a starting point. Here there are even numbers of both types of tile so 2 is a good place to start.

The tiles can then be arranged as follows:

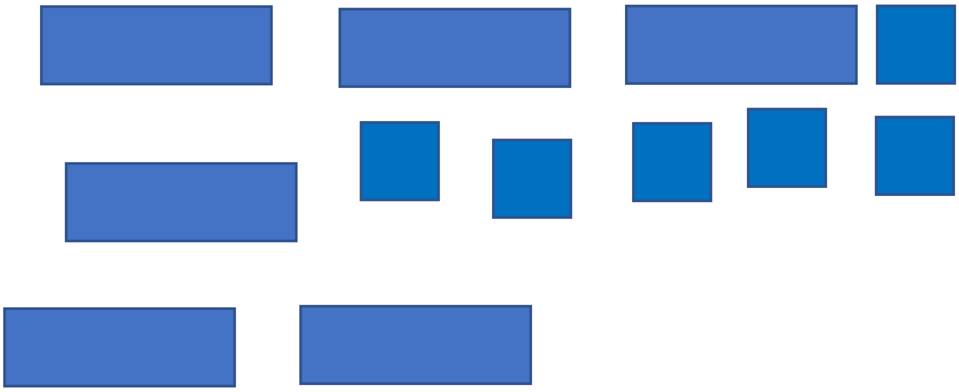


$$4x + 6 = 2(2x + 3)$$

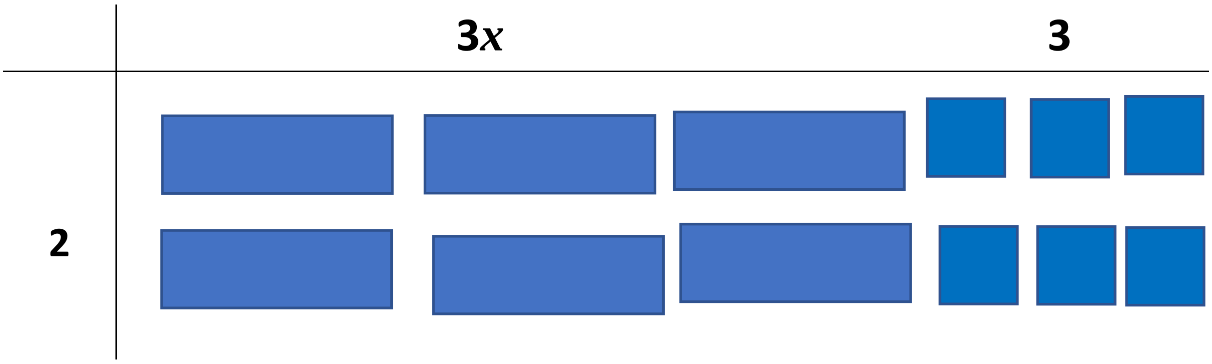
Give the students the opportunity to experiment with different arrangements. Some will find a correct one immediately, others will struggle. This exercise requires some spatial awareness.

Often there may be more than one way of arranging the tiles.

Given 6 x-tiles and 6 unit-tiles:

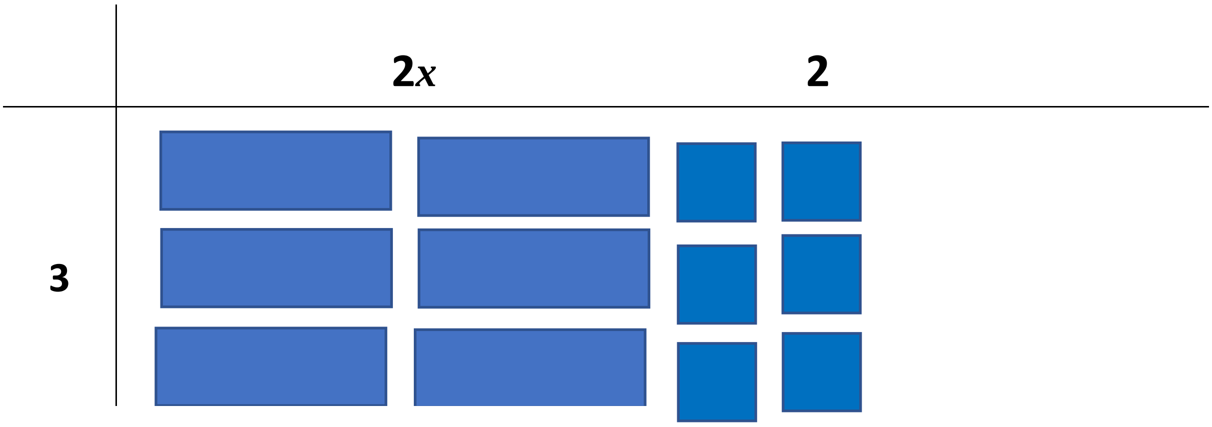


They can be arranged as:



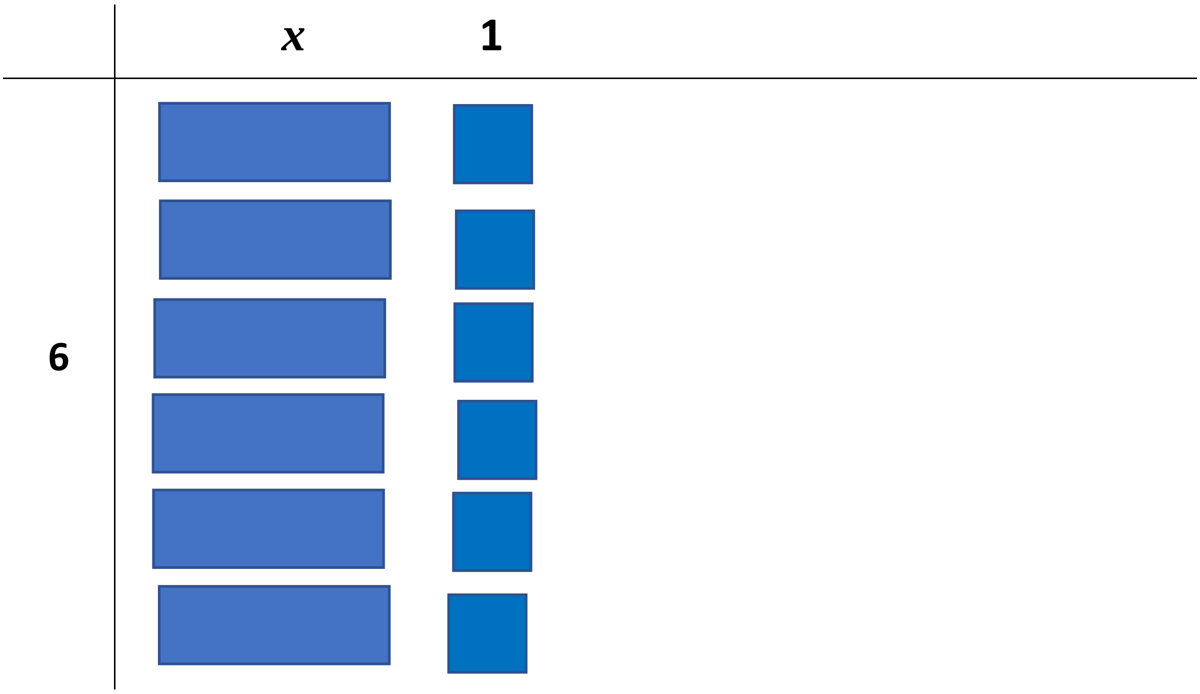
$$6x + 6 = 2(3x + 3)$$

Or as:



$$6x + 6 = 3(2x + 2)$$

Or as:

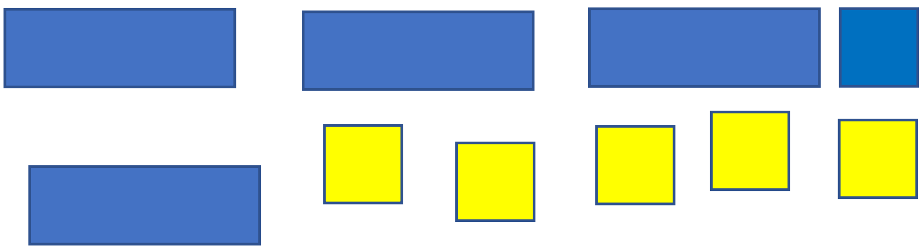


$$6x + 6 = 6(x + 1)$$

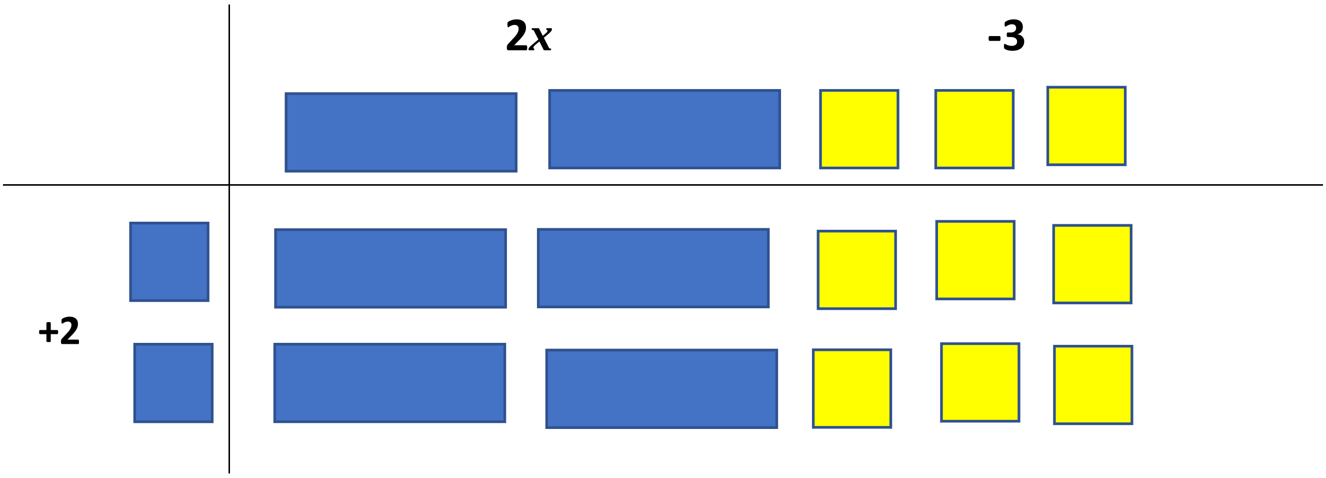
Showing these different arrangements should lead to a discussion about what is meant by “fully” factorised. Which of these arrangements is “better” and why?

You can then move on to expressions with negative numbers and/or negative factors.

e.g. If they were given the following tiles:

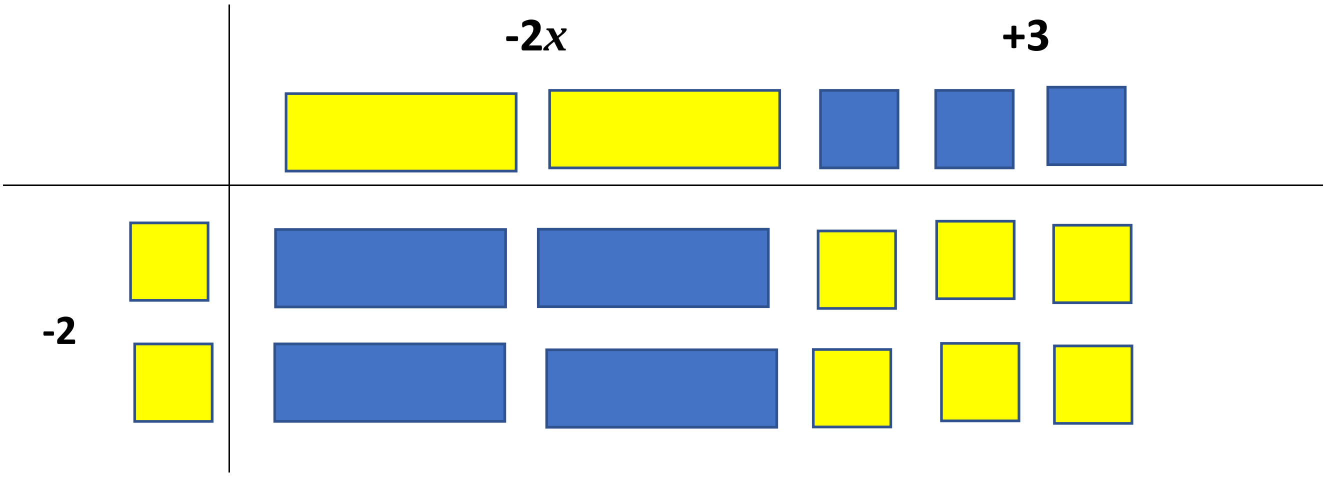


The tiles can then be arranged as follows:



$$4x - 6 = 2(2x - 3)$$

or:



$$4x + 6 = -2(-2x + 3)$$

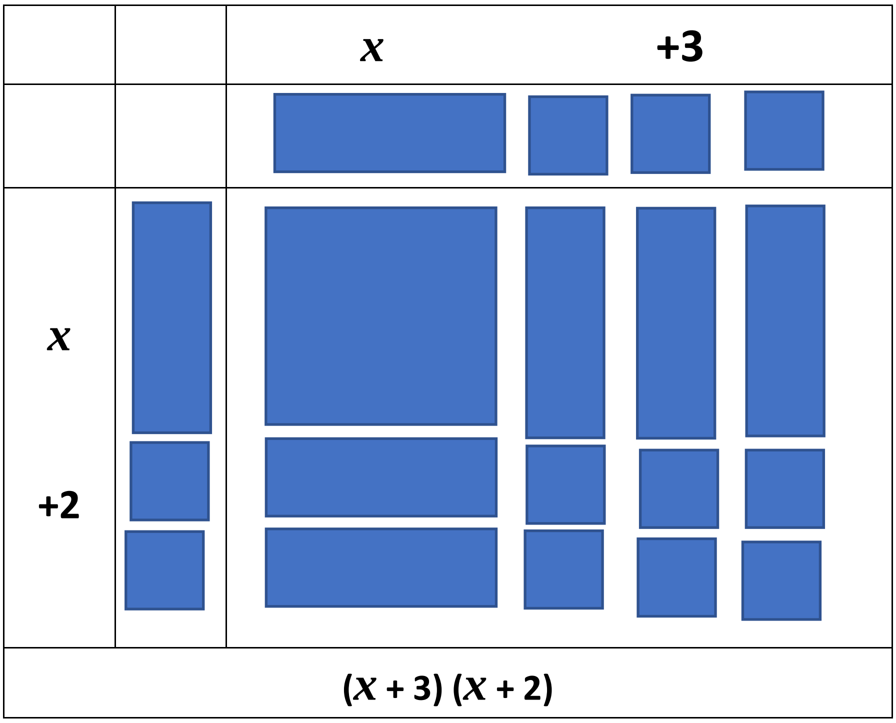
It is important for students to see that these arrangements are equivalent and that both may be useful.

Factorising quadratic expressions – with coefficient of $x^2 = 1$

Once students are confident in factorising by an integer factor you can move on to factorising quadratic expressions. Start with expressions where the coefficient of x^2 is 1.

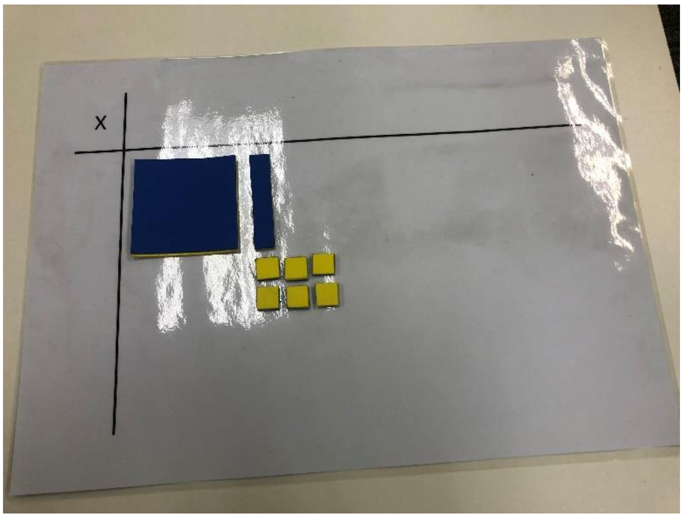
Students should now be familiar with the concept of arranging tiles into a rectangle to factorise. Finding the correct arrangement will be challenging for some students. They should be encouraged to move the tiles around until they find a rectangle.

e.g. $x^2 + 5x + 6$

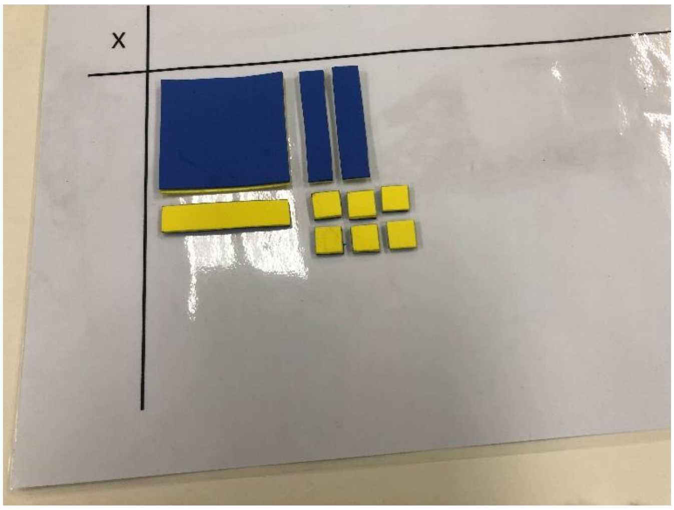


When you introduce negative tiles, you may need to add zero-sum pairs in order to form a rectangle.

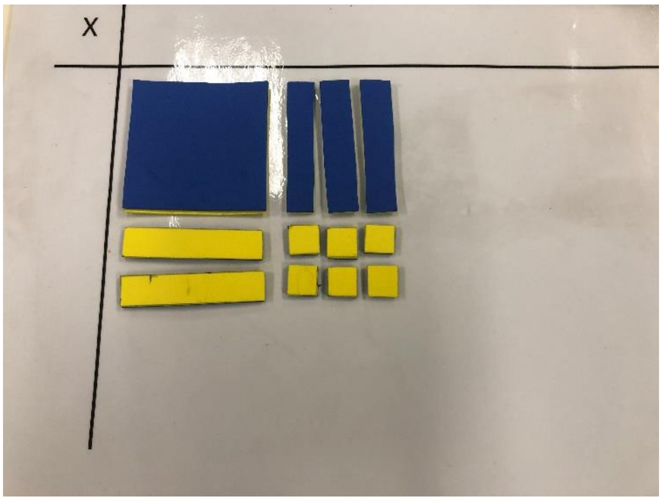
e.g. $x^2 + x - 6$, when you try and form a rectangle with these tiles you will find that it is not possible



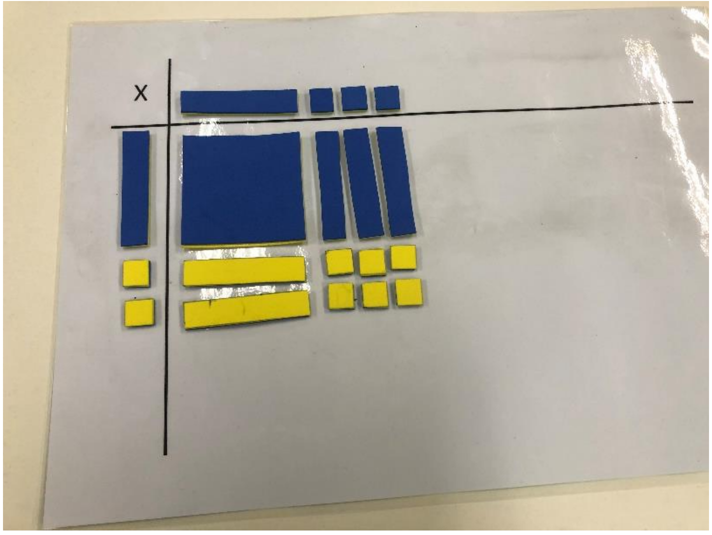
Discuss with the students how you can fill in the rest of the area.



Show how you can add zero-sum pairs, without changing the “actual” number of tiles.



Here you can see that by adding 2 lots of x and $-x$, we have been able to fill-in the rectangle.



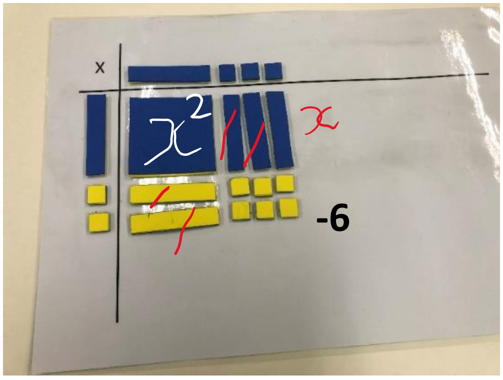
This gives us the factors (x + 3)

and (x – 2)

(It is important to remind the students that the dimensions of the x-tile are 1 by x. the x dimension is **not** a multiple of 1.)

Encourage students to relate the areas formed to the product when you multiply out the brackets.

$(x + 3)(x - 2) = x^2 - 2x + 3x - 6 = x^2 + x - 6$



Ask students which parts, the x^2, the x or the constant gives them a clue as to how to arrange into a rectangle.

Allow the students to practice and to develop their own strategies.

Encourage them to move from the concrete to representational and abstract modes.

Not all students will need to spend time on the representational mode, some will be able to move straight to abstract. Representational includes taking photographs of what they are doing to make story-boards, as I have here.

Allow the students time to discuss their strategies in groups, explaining how they came to their ideas.

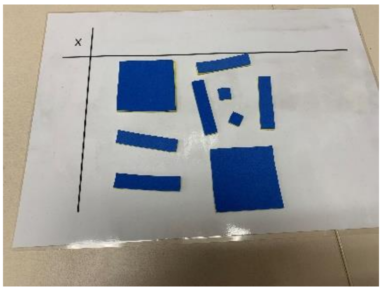
Factorising quadratic expressions – with coefficient of $x^2 > 1$

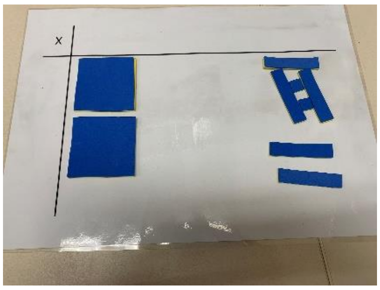
Students often struggle when trying to factorise a quadratic expression when the x^2-coefficient is greater than one. Allowing students time to 'play' with the algebra tiles to form rectangles can help them to form a mental picture of how to do this. The algorithms used can be quite cumbersome and using the algebra tiles will help students to understand why/how an algorithm works.

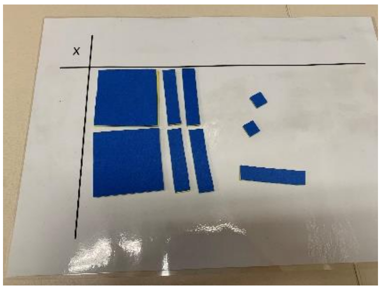
In the following examples I will explain *an* approach students can take. It is important that students are given the opportunity to work on the problems first without guidance.

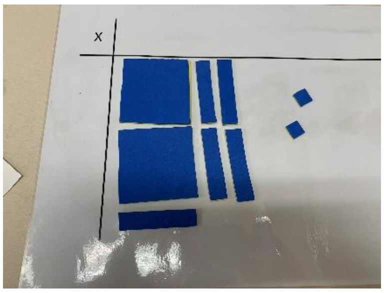
Example 1

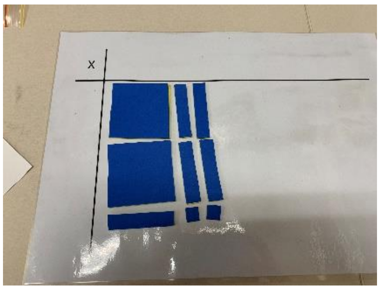
Given: $2x^2$, $5x$ and 2 units, arrange in a rectangle to find the factors.











Here I started with the $2x^2$

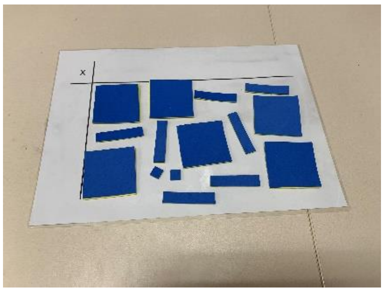
Then arranged the x around them

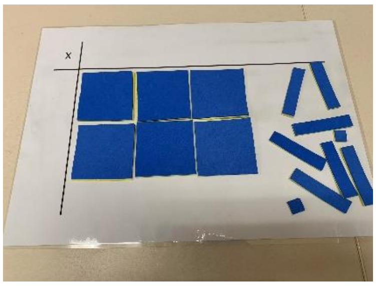
Then fitted the units in the gaps

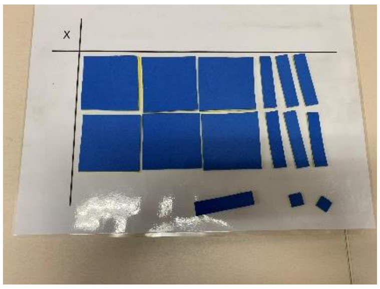
Starting with the x^2 and forming them into a rectangle is a good starting point.

Example 2

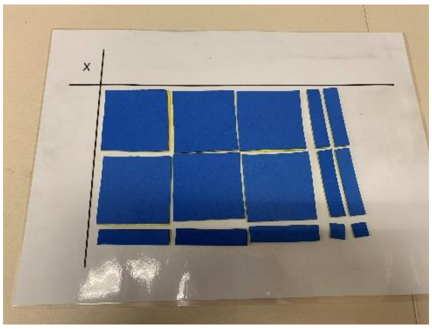
Given: $6x^2$, $7x$ and 2 units, arrange in a rectangle to find the factors.

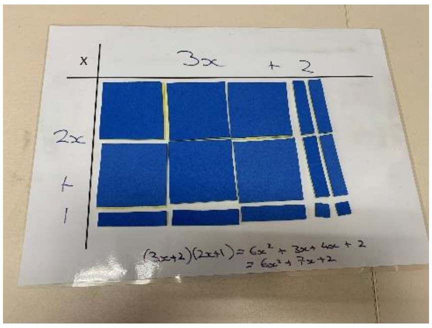






You will not always get the correct arrangement first time!





Arrange the x tiles around the x^2 tiles

Then fit in the unit tiles

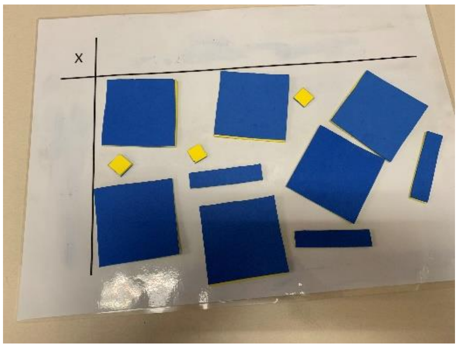
Use the laminated sheet to annotate what you do.

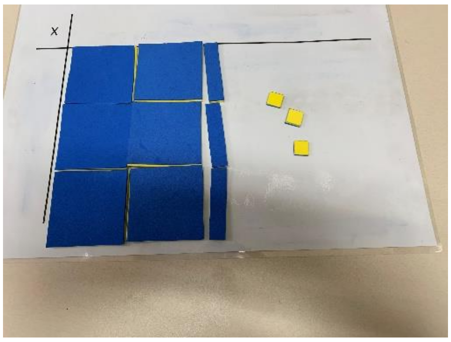
Again, encourage students to relate the various sections to the product of the brackets.

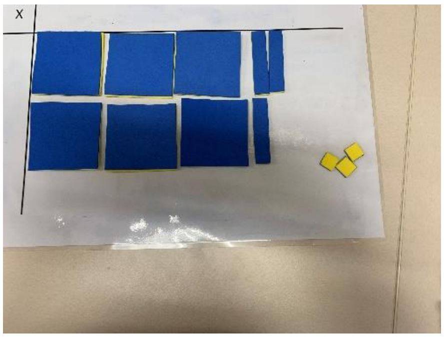
Example 3

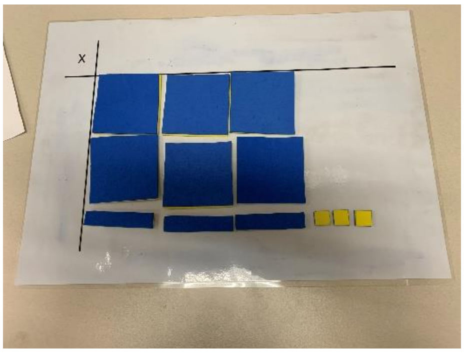
Now introduce negative coefficients.

Given: $6x^2$, $3x$ and -3 units, arrange in a rectangle to find the factors

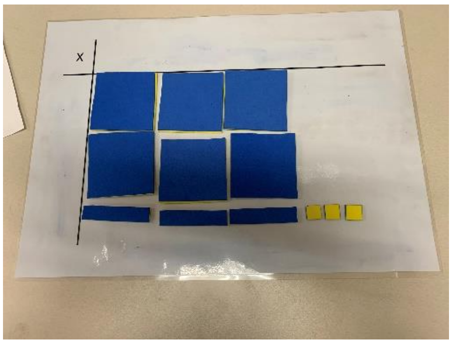


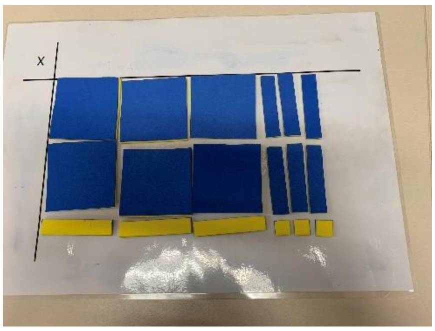


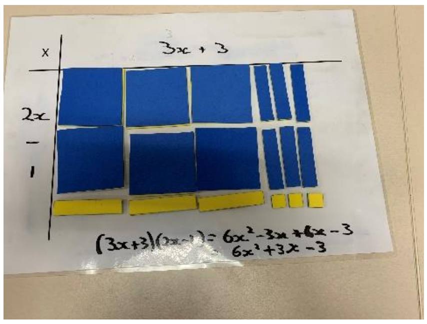




After trying a few arrangements, it should become obvious that it is not possible to form a rectangle with the given tiles. Students should, by now, realise that they need to add zero-sum pairs.







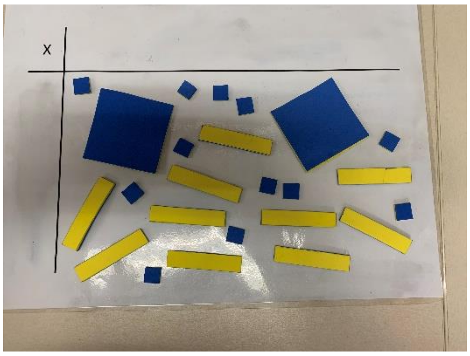
First make an arrangement that is almost rectangular.

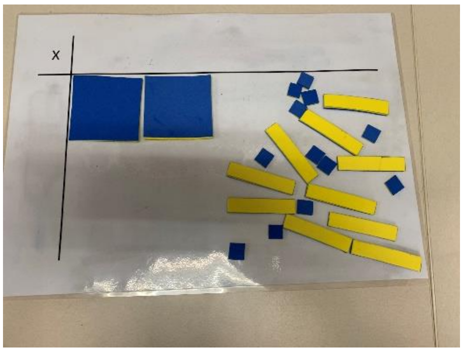
Here 3 zero-sum pairs of x-tiles have been added

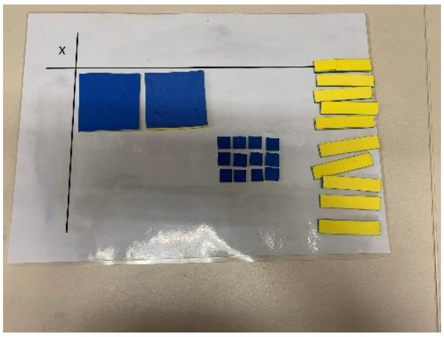
Finally, the finished rectangle has been annotated.

Example 4

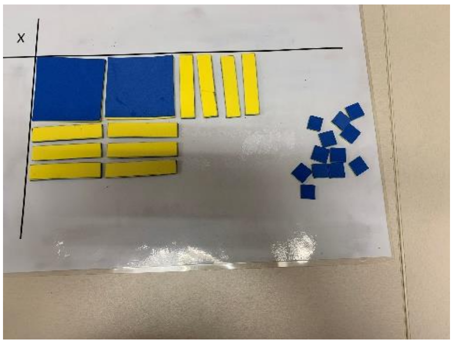
Given: $2x^2$, $-10x$ and 12 units, arrange in a rectangle to find the factors

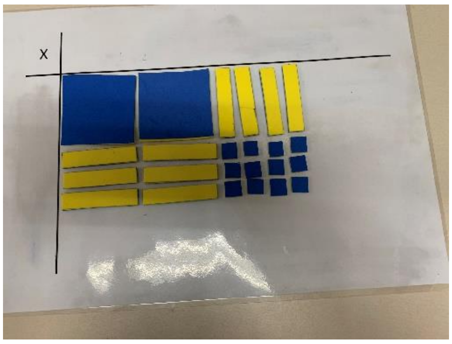




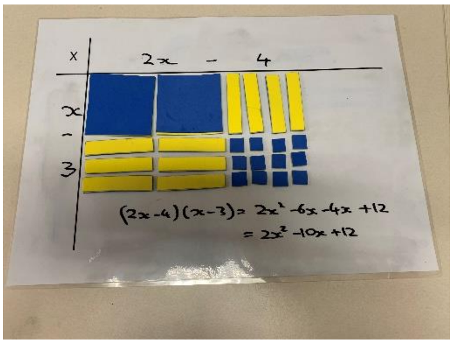


Sometimes it is easier to form a rectangle with the unit-tiles first to give a hint to the factors. This method matches the commonly used algorithm.





Remind students that a negative multiplied by a negative is positive

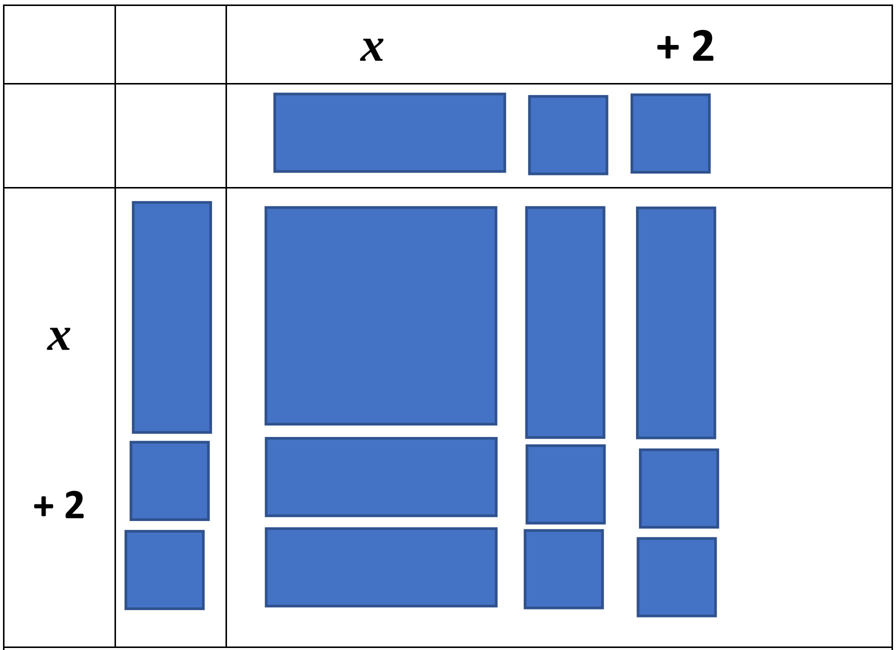


Finally, annotate to show the factors

Perfect squares - expansion

I would suggest giving the students an expansion to do using the algebra tiles and then discussing it afterwards.

Example 1: Expand the following expression $(x + 2)(x + 2)$

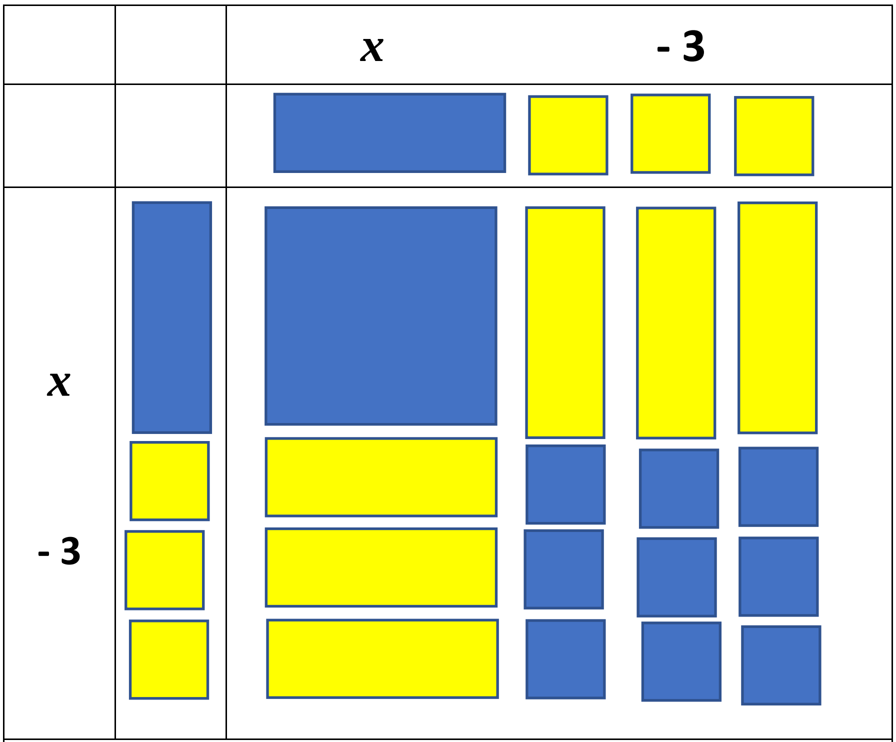


$$(x + 2)\,(x + 2) = x^2 + 4x + 4$$

Ask the students to identify what is the difference between this and the other expansions they have modelled until now. They should be able to see that this arrangement is a square. You can then discuss why this is and how else you could write the original expression i.e. $(x + 2)^2$.

Allow students to practice these expansions and see if they can deduce the identity for themselves.

Example 2: Expand the following expression $(x - 3)(x - 3)$



$$(x - 3)\ (x - 3) = x^2 - 6x + 9$$

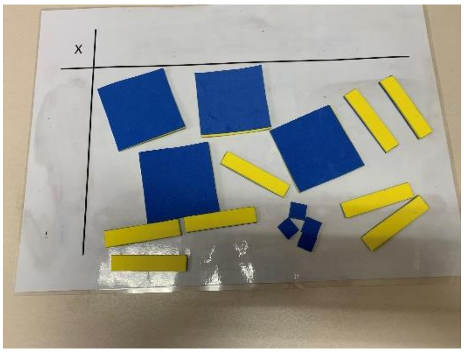
Perfect squares – factorisation

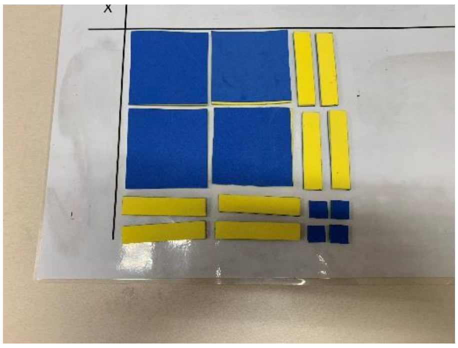
Students should be able to recognise that when they can arrange given tiles into a square the factors will be the same.

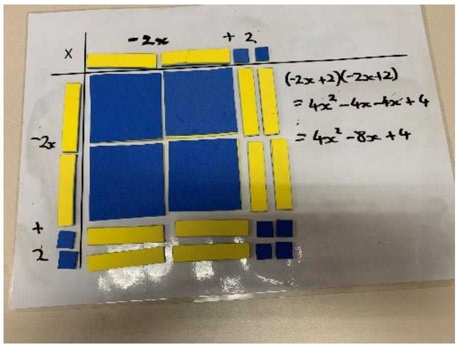
e.g. given $4x^2$, $-8x$ and 4-units, arrange in a rectangle to find the factors.

Students should see that the arrangement is a square and so the factors are the same.

(Is a square a rectangle?)







Are there other factors I could have used?

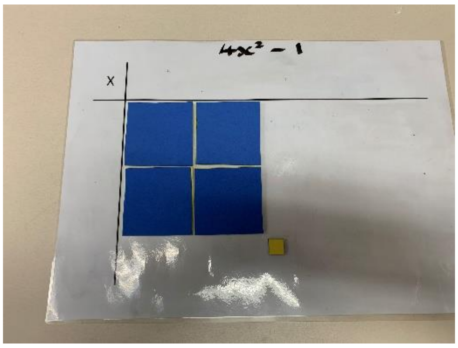
Encourage the students to identify any patterns they see.

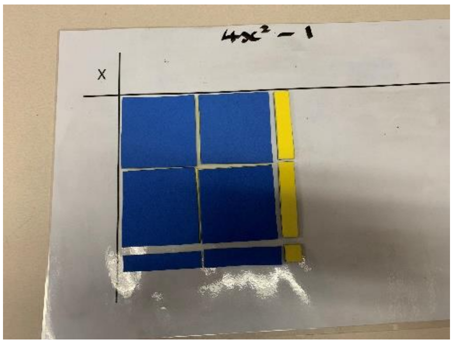
Here the x^2-tiles and the unit tiles form squares with the x-tiles filling in the gaps. Identifying these patterns will lead to finding the difference of perfect squares and completing the square.

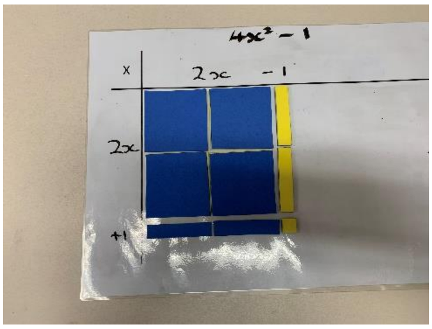
Difference of perfect squares

To find the factors of the difference of perfect squares first build the squares and then add in the ‘missing’ parts to complete the rectangle (square).

Example 1: $4x^2 - 1$





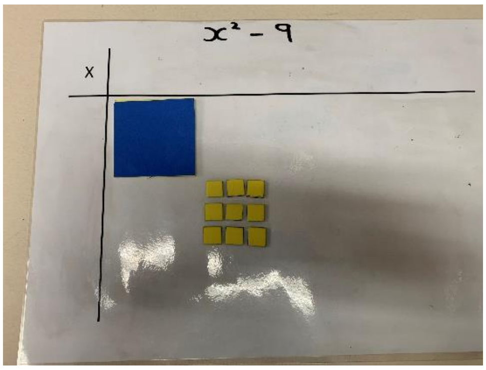


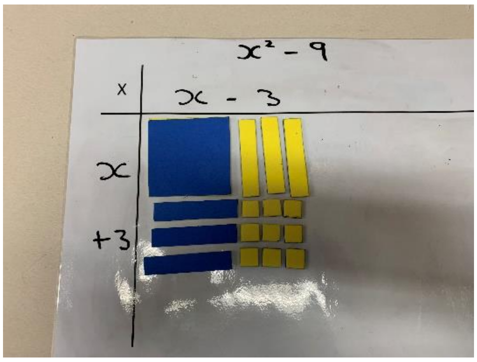
Model the question

Add zero-sum pairs to fill -in the gaps

Annotate to show the factors

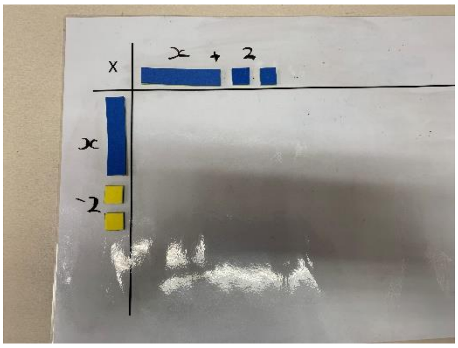
Example 2: $x^2 - 9$

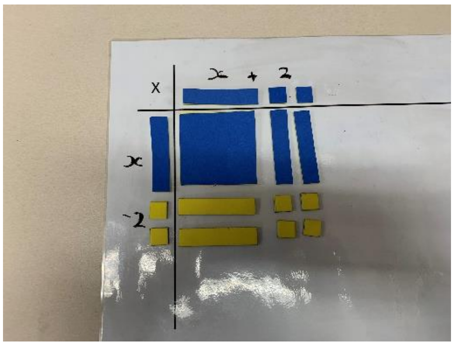


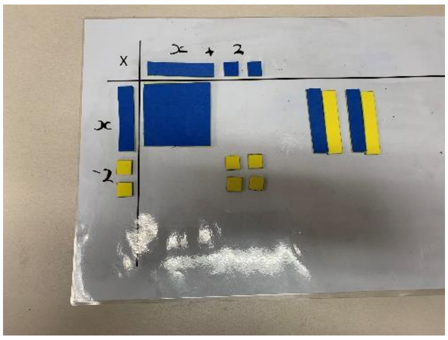


Another way of approaching this is to start with the factors and see what shapes are made.

e.g. Given $(x + 2)\ (x - 2)$







Model the question

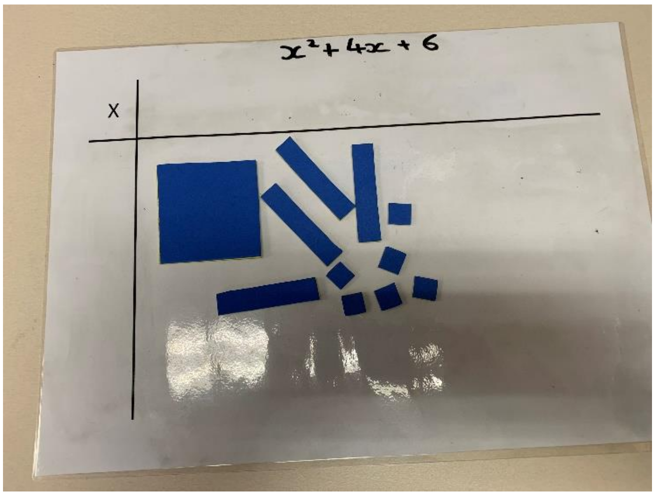
Fill in the tiles

Remove the zero-sum pairs to see what is left

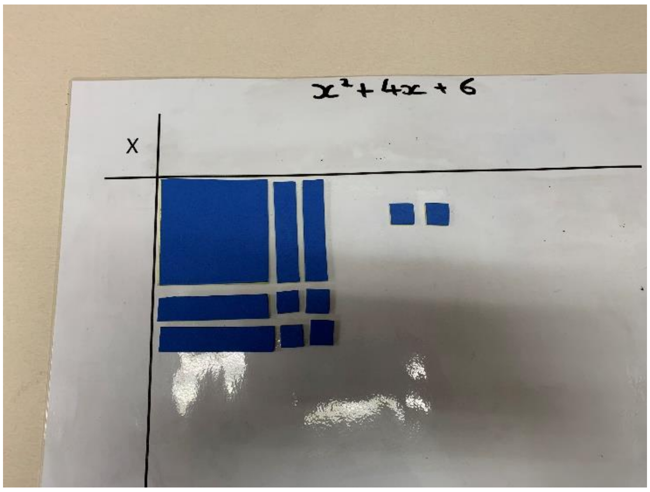
Completing the square

Completing the square can be modelled visually using algebra tiles. This is a good way for students to comprehend what is happening.

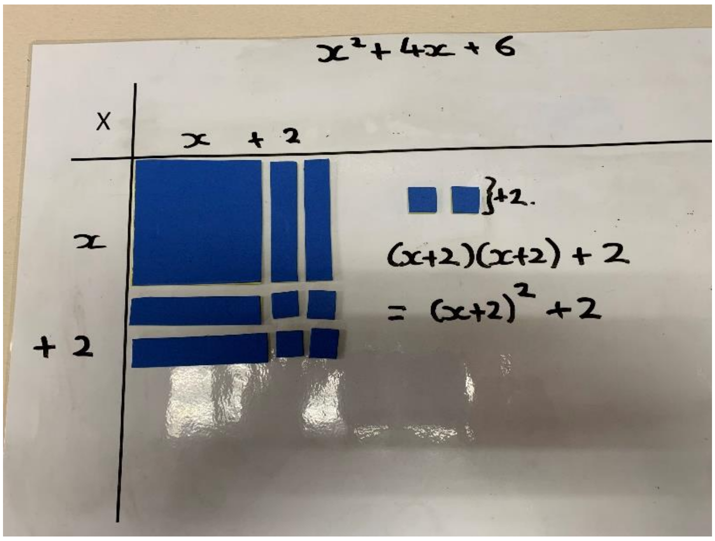
Example 1: $x^2 + 4x + 6$



Lay out the required tiles

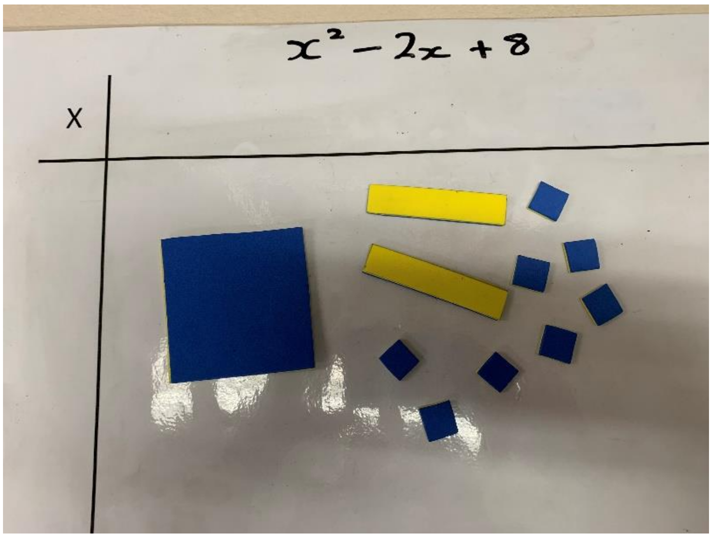


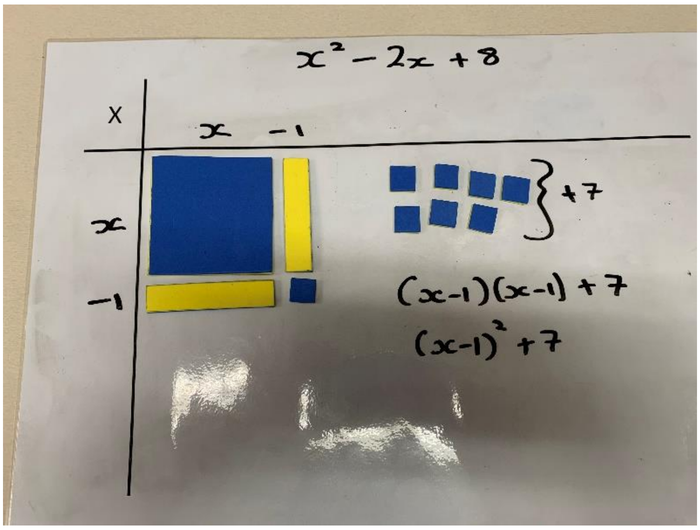
Form into a square, leaving any extra tiles to the side



Annotate to show the factors and the remaining tiles

Example 2: $x^2 - 2x + 8$



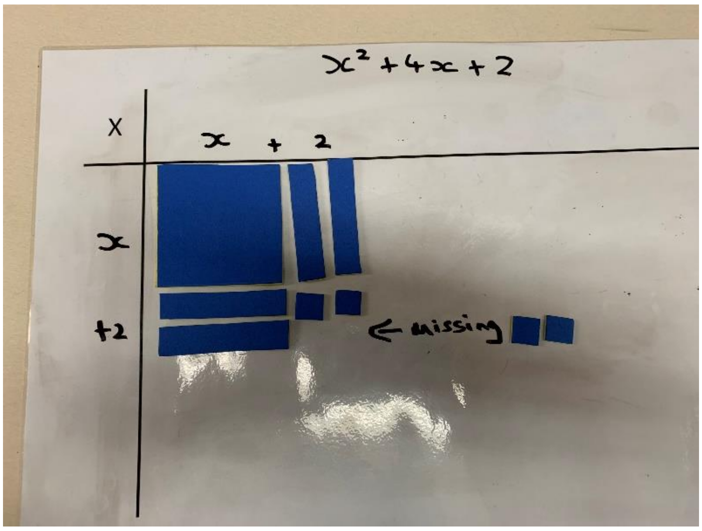


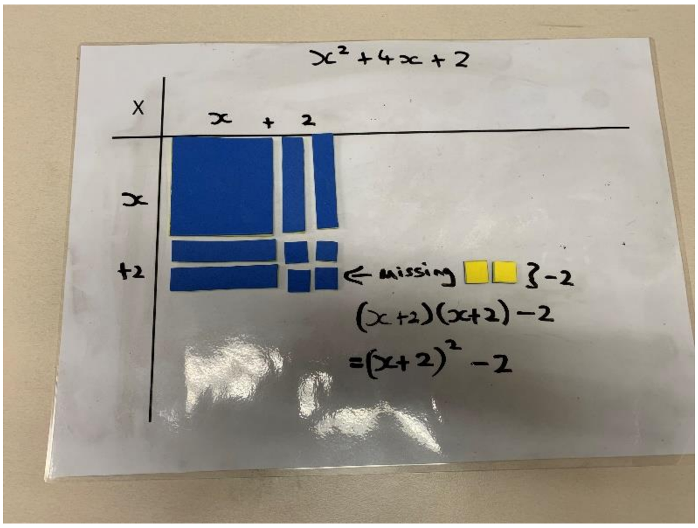
Layout the required tiles

Form into a square, leaving any extra tiles to the side.

Annotate to show the factors and the remaining tiles.

Example 3: $x^2 + 4x + 2$





Lay out the required tiles and form into a square. Here there are tiles missing.

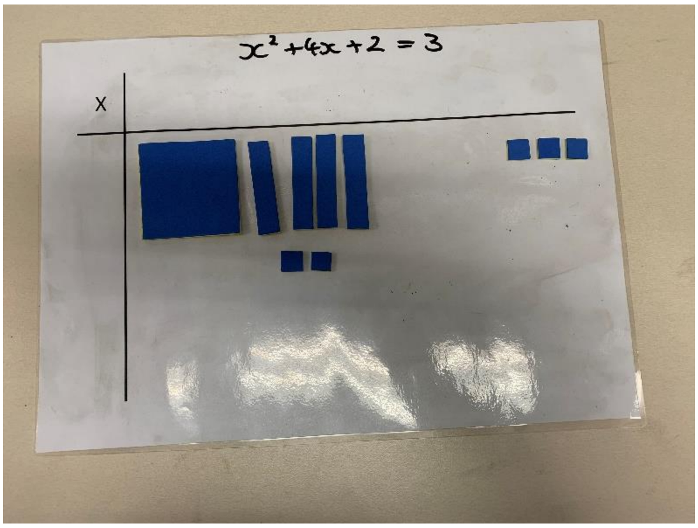
If you ‘add-in’ the missing tiles to complete the square you also must add two negative-tiles to form zero-sum pairs.

Annotate to show the factors and the tiles added, to form the zero-sum pairs, and hence balance the equation.

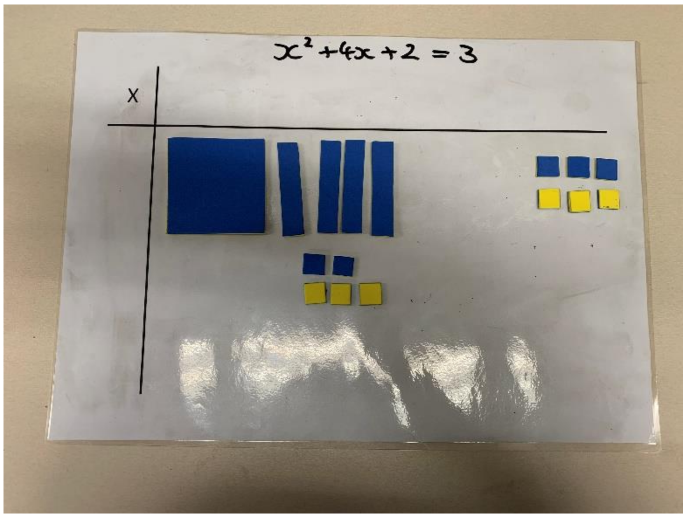
Solving algebraic equations using completing the square

Using the principles in the previous section you can then move on to solving equations using completing the square.

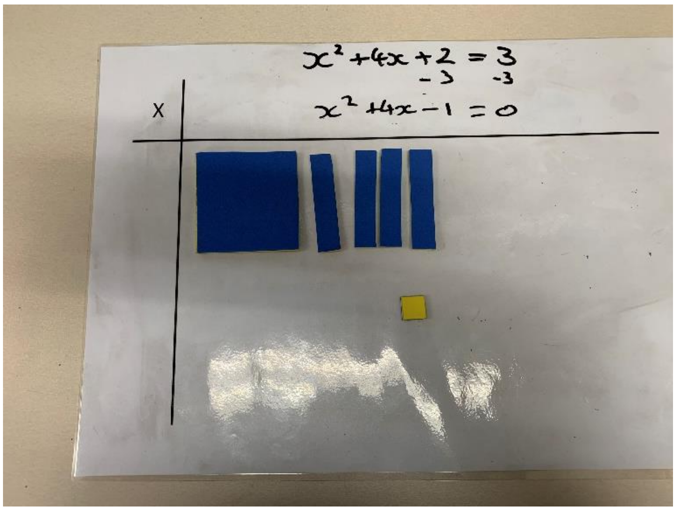
e.g. Given $x^2 + 4x + 2 = 3$



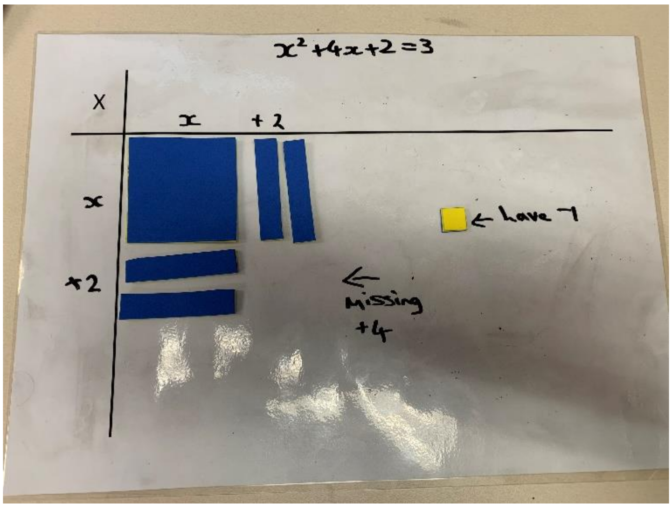
Model the problem using the algebra tiles



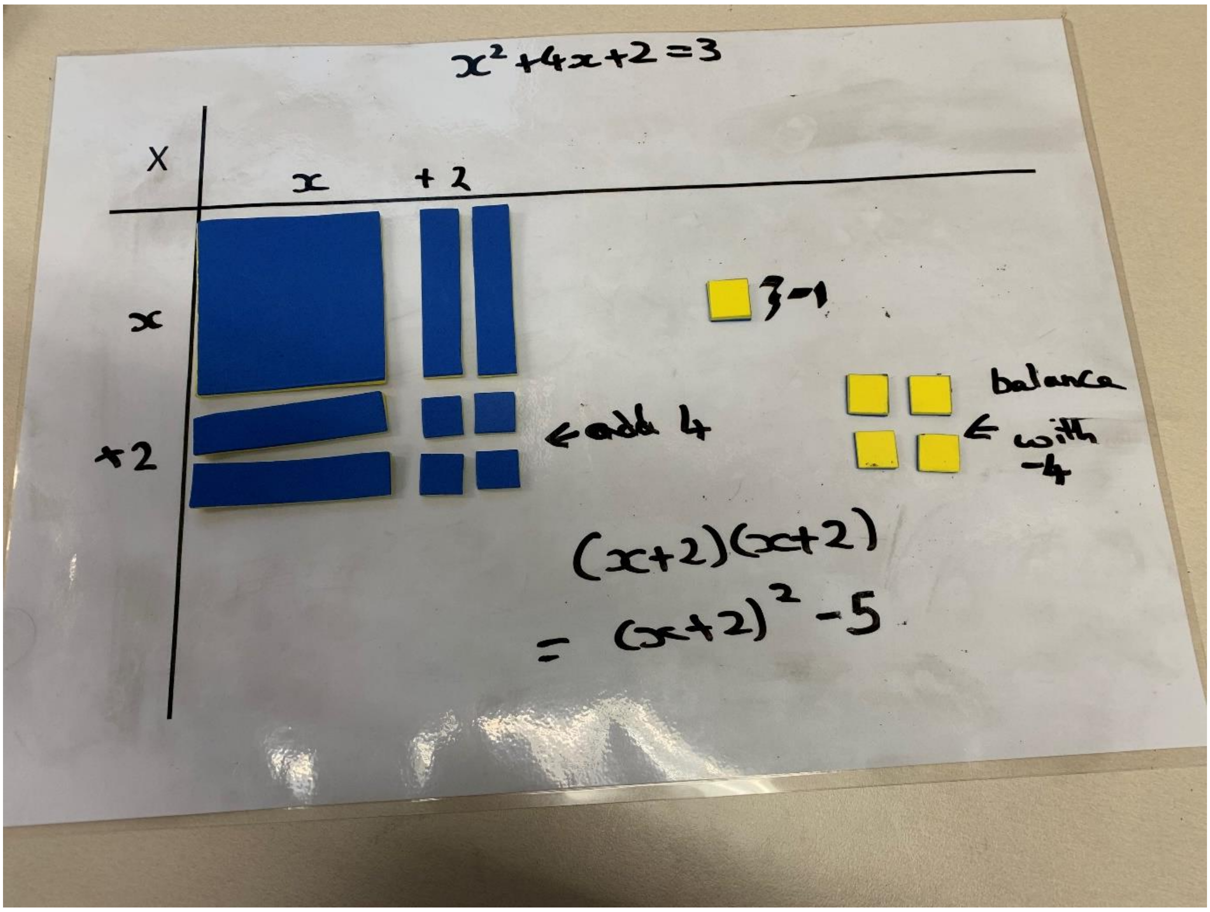
Use zero-sum pairs to set the equation equal to zero



Annotate and remove zero-sum pairs



Arrange into a square and identify what is missing (or extra)



This annotation may look a little complicated.

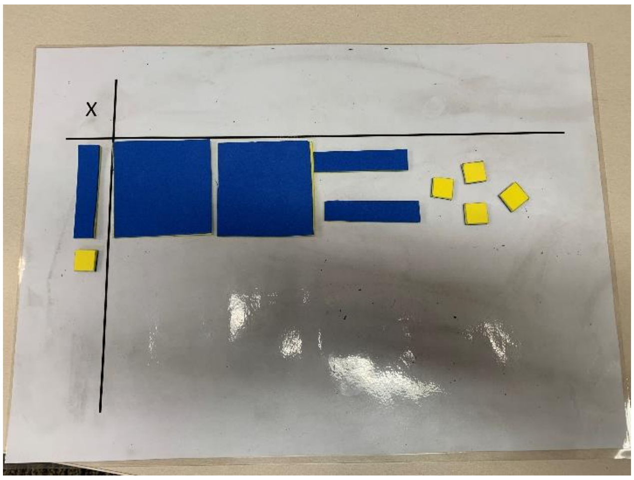
In the previous picture we identified that 4 positive-tiles were missing to form the complete square and that we had one negative tile extra.

To balance the 4 positive-tiles that must be added we also add 4 negative-tiles (i.e. 4 zero-sum pairs). This gives us 5 negative-tiles in total.

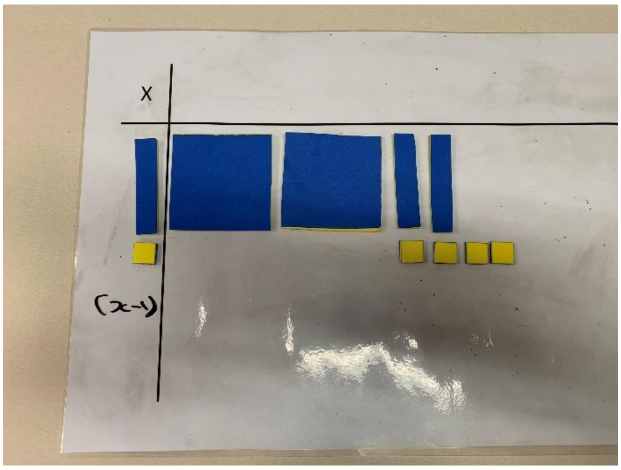
Dividing a quadratic by a linear factor

Once students are confident with factorisation you can investigate division by a given factor, as this is essentially the same idea.

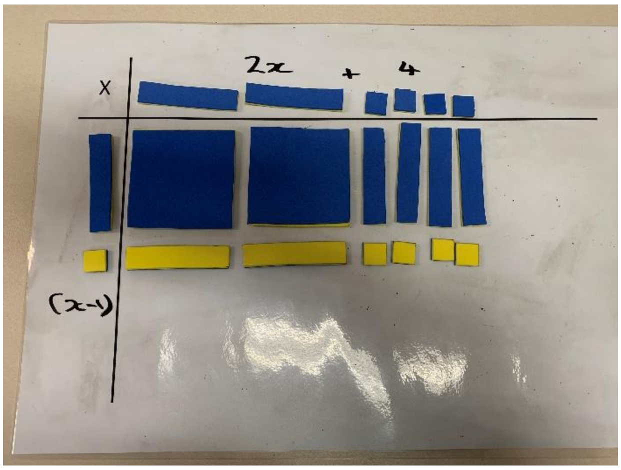
e.g. Solve $(2x^2 + 2x - 4) / (x - 1)$



Set out the tiles to match the problem



Arrange the dividend against the divisor, forming as close to a rectangle as possible. There will be 'missing' parts of the rectangle.



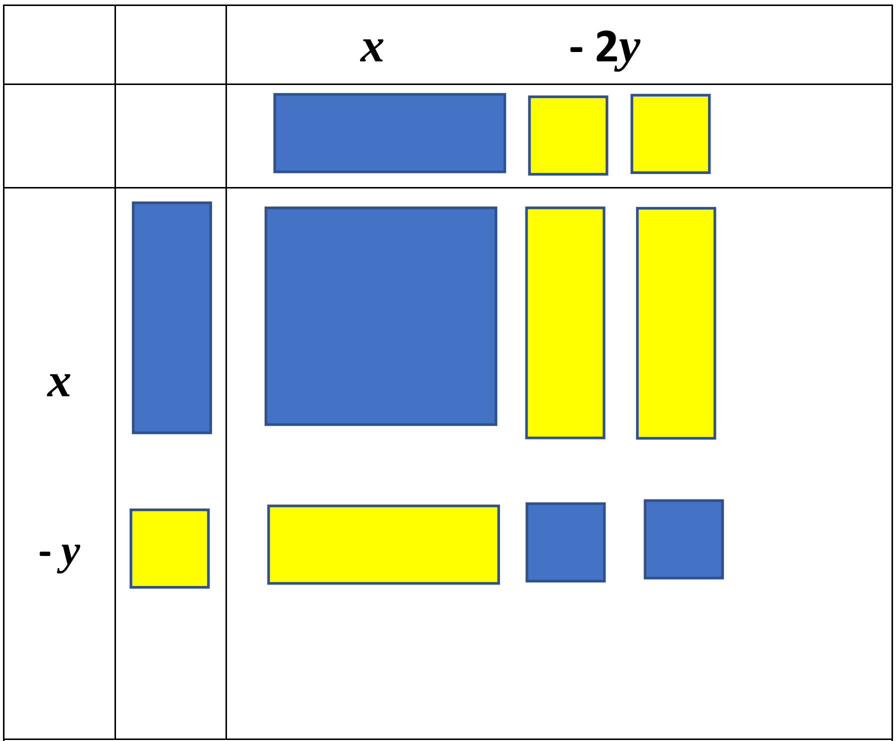
Add zero-sum pairs, as needed, to complete the rectangle and annotate.

Extending the model to x and y, 2 variables

We have said, until now, that the dimensions of the tiles are x by x, x by 1 and 1 by 1.

We can extend the model to two variables by saying the dimensions are x by x, x by y and y by y.

Example: Expand the following expression $(x - 2y)\ (x - y)$



$(x - 2y)\ (x - y) = x^2 - 3xy + 2y$

All the previous examples can then be used to explore algebra with two variables.

References

Algebra Tiles:

Working with algebra tiles by Don Balka and Laurie Boswell

Algebra Tiles Australia: A concrete, visual area-based model by Lorraine Day

Algebra Tiles Workbook, Learning resources at: https://dccmiddle.asd20.org/Teachers/Susan_Turner/Documents/algebra%20tiles%20workbook.pdf

https://www.drpaulswan.com.au/operating-theatre

https://www.aamt.edu.au/Webshop/Entire-catalogue/Algebra-Tiles-Australia-a-concrete-visual-area-based-model-book)

http://www.greatmathsteachingideas.com/2015/04/04/algebra-tiles-from-counting-to-completing-the-square/

CRA Model:

Bruner, J. S., & Kenney, H. J. (1965). Representation and mathematics learning. *Monographs of the Society for Research in Child Development*, *30*(1), 50-59.

http://www.fldoe.org/academics/standards/subject-areas/math-science/mathematics/cra-model.stml

https://www.youtube.com/watch?v=0Oa3dZCPeRM

https://www.youtube.com/watch?v=weCPBlJVSrI

Area model:

James Tanton video: https://www.youtube.com/watch?v=Sfi4QUlQ4co

Completing the square using Algebra Tiles:

https://www.youtube.com/watch?v=bpJyJlCnR3

https://www.youtube.com/watch?v=ebtjNIpFSgg