Investigating Irrational Numbers – Background Notes (Year 8)

Where are Irrational Numbers used in the real world?

Lots of places!

Irrationals like π and surds like $\sqrt{2}$ or $\sqrt{3}$ are used in geometry all the time, for working out the dimensions of two-dimensional shapes and three-dimensional objects and volumes.

Special irrational numbers like the natural logarithm 'e' (sometimes called 'Euler's Number') are used in growth functions (equations and graphs which describe the rates at which things grow or change, such as the growth of plants or the spread of a disease), as well as in the calculation of compound interest in the world of finance.

(a) 'Pi' and the world of circular geometry

The irrational number Pi (π) is used to calculate circumferences, areas and volumes of circles and cylinders. This means that π has lots and lots of uses in architecture, manufacturing, design, computer graphics… just about anywhere a circular or a cylindrical object has to be used or made.

(b) The 'Golden Ratio', art and architecture

The 'Golden Ratio', or 'Phi' (φ) is a special irrational number used frequently for working out visually pleasing proportions in art, architecture and design. It is calculated as $\frac{1+\sqrt{5}}{2}$, or 1.61803 (rounded to 5 decimal places).

The Golden Ratio often occurs in nature as well – such as in some flower and shell patterns and proportions – and the human eye seems to find design using these proportions particularly beautiful.

(c) Surds, engineering and design

Not all roots of numbers are irrational, however most are – and we all these irrational roots 'surds'. Surds (such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{7}$, $\sqrt{12}$, $\sqrt{5.25}$) are often used for calculating precise measurements in the rectangular and triangular geometries used in engineering, design and computer graphics (such as those used to create CGI movies and video games).

Also, irrational surds are often used in parabolic geometry – such as that used in satellite technologies, ballistic trajectories and the modelling and creation of curved surfaces.

(d) The 'Natural Logarithm' and Exponential Growth Curves in Nature and Finance

The Natural Logarithm 'e' – which is often called 'Euler's Number' - is a special irrational number (2.71828 to the first 5 decimal places) that is used to calculate exponential growth rates in natural phenomena like plant growth, animal population growth and the spread of microorganisms such as bacteria and viruses. The curves used to track and predict the growth of the Covid-19 virus in 2020 used the Natural Logarithm to describe its exponential growth.

e is also used in the calculation of compound interest and depreciation in finance and investment. This is because the formula for calculating e is very similar to that for calculating compound interest. The more frequent the compounding of an amount of money such as an investment along with the interest earned or paid, the closer the rate of the growth of the investment (or debt) is to the true vale of e.

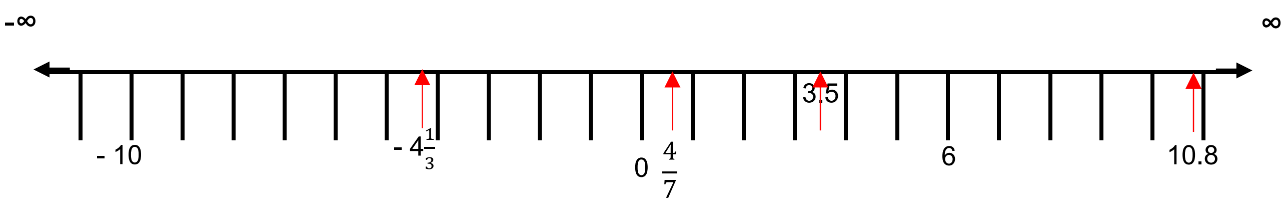
What is an Irrational Number?

In order to investigate **irrational numbers**, first we need to look again at what a **rational number** is.

There are a few ways we can think about **rational** numbers. The first is to imagine a number line that goes from minus infinity (-∞) to infinity (∞).

Any rational number is one that can be placed on a precise point somewhere on that number line. All whole numbers or integers (including negative integers) are rational numbers, as are all decimal numbers that are terminating.

Recurring decimal numbers – numbers that can also be expressed in their fraction form (e.g. ‘0.66666…’ or ‘$0.\dot{6}$’ can be written in its fraction form as ‘$\frac{2}{3}$’) – are also rational, because their fraction can also be easily placed on a precise point on the number line.



Another way of thinking about **rational** numbers is this:

If you can express the number as a fraction (with a numerator and denominator that are both integers), then the number is rational.

Of course, all integers can be expressed as fractions; the number ‘6’ can be written as ‘ $\frac{6}{1}$ ’, the number ‘-54’ can be written as ‘ $\frac{-54}{1}$ ’, and so on.

Notice that ‘6’ can also be written as ‘ $\frac{36}{6}$ ’, or as ‘ $\frac{54}{9}$ ’… actually, in a whole heap of different ways. The point is that all of these ways of expressing ‘6’ occupy the exact same point on the number line – at precisely and exactly the rational number ‘6’.

All recurring decimal numbers, when expressed in their fraction form, are also rational *because we can locate a fraction at a precise point on the number line*. The number ‘$6.\dot{4}2857\dot{1}$’ is considered **rational** (even though the decimal digits repeat forever) because as a **recurring** decimal we can write this number **precisely** as the fraction ‘$6\frac{3}{7}$’.

So… what is an **‘irrational number’**?

Well, actually not all numbers can be expressed in a fraction form (in other words, as a ratio between two integers).

Some numbers are used in the real world for important calculations, but **we can’t actually write them in a precise way** other than using some special mathematical notation (symbols) to represent them.

In fact, a simple definition for an irrational number is:

An irrational number is a real number that can’t be written as a ratio or fraction between two integers.

The best known examples of irrational numbers are:

- π ('Pi') – *approximated* by 3.141592653589793… (and more, forever…);
- $\sqrt{2}$ ('The square root of 2') – which is a *surd*. Surds are irrational roots of rational numbers. $\sqrt{2}$ is approximated by 1.41421356237… (and more, forever…);
- φ (The 'Golden Ratio', or 'Phi') - *approximated* by 1.618033988… (and more, forever…); and
- e ('Euler's Number' or the 'Natural Logarithm') - *approximated* by 2.718281828… (and more, forever…)

What do you notice about the decimal approximations for all of these numbers?

Let's take a look at a few of these numbers!

π ('Pi') is a number used for the calculation of circumference, area and volumes of circular shapes and cylindrical and spherical objects. As a number, Pi can be calculated as **the ratio between the circumference and the diameter of any circle**: $\pi = \frac{\text{Circumference}}{Diameter}$ = 3.141592…

You can test this yourself. Grab any circular object – e.g. a plate out of the kitchen cupboard – and use a flexible tape measure (such as a dressmaker's tape) to accurately measure across the circle, making sure your line of measure passes through the dead centre. Then, use the same measuring tape to take a measure around the outside or circumference of the circle. Divide the circumference measure by the diameter measure. What do you get? That's right… Pi (or close to it)!

Try any different sized circular object… the result will be the same! Wow!

Let's now look at some surds.

We need first to be aware that some roots of rational numbers are also actually **rational**, as they can be simplified into an integer or a number that can otherwise be expressed as a fraction.

For example, roots of numbers that are the result of multiples of a single rational number – such as '$\sqrt{4}$' *(= 2)* or '$\sqrt{9}$' *(= 3)* or '$\sqrt[3]{125}$' *(= 5)* or even $\sqrt[4]{2{,}762.81640265}$ *(= $7\frac{1}{4}$)* are **not** irrational, as we can simplify them into a rational number (i.e. a number that can be expressed as a fraction).

But… **most roots are irrational** because they can't be simplified into a rational number (or fraction) form. This is because their numerical expression ends up as a **non-terminating**, **non-recurring** decimal number. We call this large group of irrational roots 'surds'.

Examples of surds are:

$\sqrt{2}$; as a decimal, this is expressed as 1.4142135623… (and on and on…)

$\sqrt{3}$; as a decimal, this is expressed as 1.7320508075… (and on and on…)

$\sqrt[3]{125}$; as a decimal, this is expressed as 2.15443469003… (and on and on…)

… and many more!

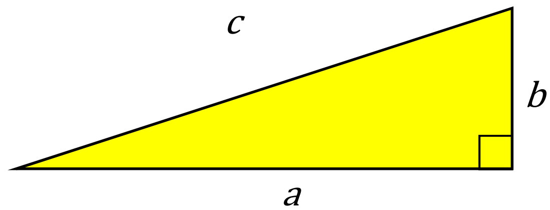
Let's take a look at how surds, such as $\sqrt{2}$ and $\sqrt{13}$, for example – are used in practical mathematical contexts.

To do this we'll take a detour into the work of Pythagoras, who tells us how to calculate the hypotenuse of a right angled triangle:

Base side length = a

Vertical side length = b

Hypotenuse = c



Pythagoras's Theorem tells us that the squared value of the hypotenuse is equal to the sum of the squared values of the other two sides of a right triangle.

So, $c^2 = a^2 + b^2$. So, rearranging, this means that $c = \sqrt{(a^2 + b^2)}$.

If we put some numbers against these values, we can see why many surds are irrational.

Fore example, if we let $a = 3$ and $b = 2$, then:

$$c = \sqrt{(a^2 + b^2)}$$
$$= \sqrt{(3^2 + 2^2)}$$
$$= \sqrt{(9 + 4)}$$
$$= \sqrt{13}$$
$$\approx 3.60555127\ldots$$

You'll notice that this decimal number (the value of c, that is, the length of the hypotenuse of this triangle) is both non-recurring (no repeating pattern) and non-terminating (it never ends). In other words… it's irrational!

Properties of Irrational Numbers

When discussing the 'properties' of a number, we are referring to what the number 'does' – or what mathematical results we get – when we operate with this type of number.

(Remember, a mathematical operation refers to any kind of change or function made to the number, such as adding to or subtracting from the number; multiplying or dividing by; squaring or raising the number to another power; finding a root of a number; and so on.)

Let's take a look at some of those operational properties of irrational numbers.

The first important property to consider for irrational numbers is that, when expressed in their numerical decimal form, they will result in a **non-terminating, non-recurring decimal number**. The examples of π, $\sqrt{2}$, φ and e are all examples that have been discussed earlier:

- When π is simplified, we take the circumference (c) of any circle and divide it by the diameter (d) of the circle, and we always end up with $\frac{c}{d}$ = 3.141592… *- which is non-terminating and non-recurring.*

- When $\sqrt{2}$ is simplified into a numerical form, we get 1.4142135623… - *which is non-terminating and non-recurring.*
- When φ is simplified from its formulaic representation $\frac{1+\sqrt{5}}{2}$, we get 1.680339… - *which is non-terminating and non-recurring.* (Note that this was inevitable, since the term '$\sqrt{5}$' in the numerator of the Golden Ratio is itself irrational!)
- When e is simplified from its representation of $(1 + \frac{1}{n})^n$, for any number value of n, we get a *non-terminating, non-recurring decimal number* that approximates 2.718281828459045235… (In fact, the larger the value of n, the closer the result of Euler's formula to the 'real' value of e!)

A second important property specific to **surds** is that **for any given number n, if n is NOT a perfect square, then $\sqrt{n}$ is going to be irrational**.

For example, the number '16' is a perfectly square number, as 4 x 4 = 16. Hence, $\sqrt{16}$ = 4, which is rational.

In contrast, the number '17' is NOT a perfectly square number. The square root of 17 will be irrational; $\sqrt{17}$ = 4.1231056…, which is irrational.

This property of course extends to roots other than square roots.

Consider cube roots ($\sqrt[3]{\ }$) of numbers. For any given number n, if n is NOT a perfect cube (or 'cubic number' – such as 8, 27, 64…), then $\sqrt[3]{n}$ is going to be irrational. Thus, the square root of 4 ($\sqrt{4}$) is rational (it's '2'), but the cube root of 4 ($\sqrt[3]{4}$) is irrational (it's 1.5874011…).

What about 'root 4' ($\sqrt[4]{\ }$) of numbers? In this case, for any given number n, if n is *not* the result of a rational number x raised to the power of 4 such as 16, 81, 256… , then $\sqrt[4]{n}$ is going to be irrational:

If $n \neq x^4$ where x is rational, then $\sqrt[4]{n}$ is irrational.

And we could keep going, extending this property to surds that are $\sqrt[5]{\ }$, $\sqrt[6]{\ }$ and so on. You can see the pattern can't you?

We can generalise this rule or property like so:

For any given number n, if n is not the result of a rational number x raised to the power of i (i.e. $n \neq x^i$), then $\sqrt[i]{n}$ is going to be irrational:

If $n \neq x^i$ where x is rational, then $\sqrt[i]{n}$ is irrational.

Finally, we can consider what happens when we **add, subtract, multiply or divide irrational numbers**, either with themselves OR with one or more rational numbers. The rules that apply here are exactly the same as they are in algebra, and we will explore those rules in a further unit.

However, if you're super keen, watch the following 'Mash-Up Math' video on YouTube. Start at the 3 minute 20 second mark, and explore 'Operations with Irrational Numbers' in greater detail!

https://www.youtube.com/watch?v=RPVu3pYDUFI