SELF-SIMILAR SOLUTIONS TO THE SURFACE DIFFUSION FLOW

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1. What is Surface Diffusion Flow?

Surface Diffusion Flow describes a particular kind of surface evolution over time. The work in this project pertained to surfaces in $\mathbb{R}^2$, which are merely curves; thus, we shall refer to Surface Diffusion Flow as Curve Diffusion Flow, or CDF.

Consider a closed curve evolving over time. That is, we have some curve $\gamma(u,t) : S^1 \times \mathbb{R} \to \mathbb{R}^2$, that will evolve smoothly under the time component t. The formal definition of CDF describes when an evolving curve is evolving under CDF.

Definition 1.1. Let $\gamma(u,t) : S^1 \times \mathbb{R} \to \mathbb{R}^2$ be a series of curves. We say γ evolves under CDF if:

$$\frac{\partial \gamma}{\partial t}(u,t) = -\kappa_{ss}(u,t)\nu(u,t),$$

where κ_{ss} is the second derivative of curvature with respect to arc length, and ν is the unit normal to γ.

Another way to think about CDF is to ask the question, given a curve γ in $\mathbb{R}^2$, how will γ evolve under CDF? That is, define the curves parametrised by time to be those which will satisfy the CDF condition. This view point will be of importance to us in the next section, but ultimately it is a complicated question to ask.

To see the complications of the CDF, merely consider the ellipse paramatrised by $(2\cos(u), \sin(u))$. Although this is a relatively simple curve, one can compute the curvature and the arc-length derivatives of curvature (κ_{ss}) to see that this flow is actually quite complex.

The graph of $\kappa_{ss}(u)$ for $u \in [0, 2\pi]$ is pictured below.

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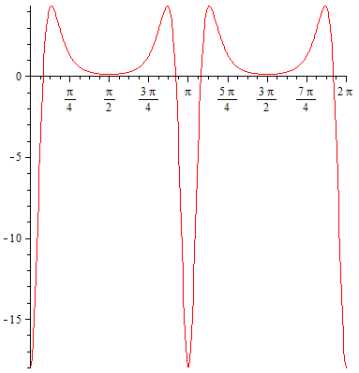
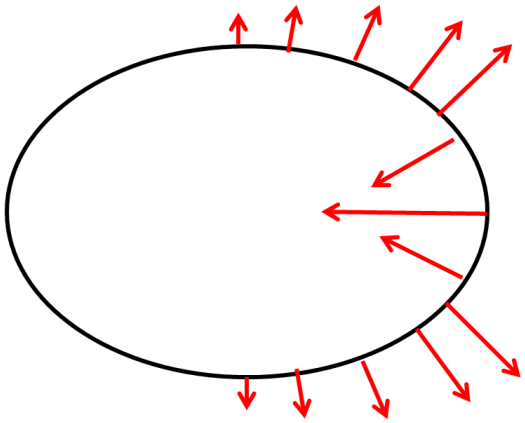


Thus, if we tried to flow this particular ellipse under CDF, remembering that we are applying a velocity of κ_{ss} in the normal direction to each point, the first time step would be represented by the following.



It can be seen that it is quickly becoming hard to picture what will happen to this curve as the curvature at each point will change quite rapidly; approximating the following timesteps will be difficult.

So if this flow is unpredictable and seemingly unintuative, why would we want to study it? The physical motivation for this flow, is seemingly endless.

Surface Diffusion Flow was first described by W. W. Mullins in his 1956 paper "The Theory of Thermal Grooving". In this paper, it is shown that SDF is a reasonal approximation for the way surface grooves develop at the grain boundaries of heated crystal structures. Another common example is the way SDF is related to the Cahn-Hilliard equation, which describes phase seperation and coarsening in the quenching process of binary alloys.

CDF has also been used to study more classical problems. Any problem in which there is some notion of volume being preserved, whilst surface area is being minimised,

can be applied to CDF. However, the motivations do carry much further than physical problems. Application into other areas of mathematics id beneficial, although unexplored in this project.

To try to understand CDF, we want to understand the second viewpoint mentioned above. That is, can we predict what will happen to a curve once CDF is applied to it. To try and get a picture of these, we look for what are called self-similar solutions, which leads us onto the next section.

2. What are self-similar solutions?

A self-similar solution to the CDF is a solution to the CDF that maintains the same shape as it evolves. That is, the curve under the flow will simply evolve by any combination of scaling, rotating or translating.

The main focus of this project was of self-similar solutions of the scaling variety. The mathematical defintion is as follows,

Definition 2.1. Let $\gamma(u,t) : S^1 \times \mathbb{R} \to \mathbb{R}^2$ be a solution to the CDF. We say that γ is a self-similar solution if:

$$\gamma(u,t) = f(t)\gamma_0(u)$$

for some function $f(t)$ and initial curve $\gamma_0(u)$.

The current knowledge of any kind of self-similar solutions to the CDF is very limited. In the 1996 paper "Curves and surfaces of least total curvature and forth-order flows" by A. Polden, it is shown that any curve with zero winding number will either develop a singularity or shrink to a point in finite time. The 1998 paper, "On the Surface Diffusion Flow" by J. Escher et al, numerically estimates the flow of a figure-8 shrinking to a point. Amazingly the numerical estimation looks like a self-similar curve, however no explicit parametrisation was proposed.

A simple and somewhat trivial example of a self-similar solution to the CDF is the circle.

Proposition 2.2. *Let $\gamma(u,t)$ be defined by:*

$$\gamma(u,t) = (A\cos(u), A\sin(u)).$$

Then $\gamma(u,t)$ is a self-similar solution to the CDF.

Proof. To see that $\gamma(u,t)$ is self-similar is trivial, since it is unchanging with time, we can simply consider:

$$\gamma(u,t) = f(t)\gamma_0(u)$$

with $f(t) = A$, a constant function, and $\gamma_0 = (\cos(u), \sin(u)$.

To see that $\gamma(u,t)$ evolves under CDF, we simply note that it satisfies the CDF equation, as:

$$\frac{\partial \gamma}{\partial t}(u,t) = \frac{\partial}{\partial t}(A\cos(u), A\sin(u)) = 0.$$

and

$$\kappa(u,t) = \frac{1}{A} \implies \kappa_{ss}(u,t) = 0.$$

Hence, the CDF equation is trivially satisfied, and the circle is a self-similar solution. □

Using this method to calculate whether or not a curve is evolving self-similarly relies on the fact that we have a parametrisation of the curve evolving under CDF. That is, we have an explicit formula of $\gamma(u,t)$. As was mentioned in Section 1, we often think of the problem as, given a curve $\gamma(u)$, how will it evolve under CDF. We now extend this question to, given a curve $\gamma(u)$, can we check whether under CDF it will evolve self-similarly? Using the next quick lemma, we can.

Lemma 2.3. *Let $\gamma(u) : S^1 \to \mathbb{R}^2$ be a curve. Then $\gamma(u)$ is a self-similar solution to the CDF if and only if:*

$$\kappa_{ss}(u) = K\langle \gamma(u), \nu(u)\rangle$$

for all $u \in S^1$.

Proof. To see this is true, let $\gamma(u,t)$ be a curve evolving under CDF such that:

$$\gamma(u,t) = f(t)\gamma_0(u).$$

We therefore have,

$$\frac{\partial}{\partial t}\gamma(u,t) = f'(t)\gamma_0(u).$$

Since $\gamma(u)$ evolves under CDF, it satisfies the CDF equation, that is:

$$\frac{\partial}{\partial t}\gamma(u,t) = -\kappa_{ss}(u,t)\nu(u,t).$$

Thus we have:

$$\begin{aligned} & f'(t)\gamma_0(u) = -\kappa_{ss}(u,t)\nu(u,t) \\ \implies & \langle f'(t)\gamma_0(u), \nu(u,t)\rangle = \langle -\kappa_{ss}\nu(u,t), \nu(u,t)\rangle \\ \implies & f'(t)\langle \gamma_0(u), \nu(u,t)\rangle = -\kappa_{ss}\langle \nu(u,t), \nu(u,t)\rangle = -\kappa_{ss}(u,t) \end{aligned}$$

Since we do not want this as a function of t, we let t_0 be such that:

$$\gamma(u,t_0) = \gamma_0(u) \iff f(t_0) = 1.$$

We finally have:

$$f'(t_0)\langle \gamma_0(u), \nu_{\gamma_0}(u)\rangle = -(\kappa_{\gamma_0})_{ss}(u).$$

□

Example 2.4. We can now use the above Lemma to check the already established self-similarity of a circle.

Consider a circle of any radius, that is $\gamma_0(u) = (A\cos(u), A\sin(u))$. Then we have:

$$\kappa(u) = \frac{1}{A} \implies \kappa_{ss}(u) = 0.$$

Thus,

$$\kappa_{ss}(u) = K\langle\gamma(u), \nu(u)\rangle$$

is trivially satisfied with $K = 0$.

Using the above result comes the next result. This result is a non-trivial, analytic self-similar solution to the CDF, previously unknown.

Proposition 2.5. *Let* $\gamma(u) : S^1 \to \mathbb{R}^2$ *defined by*

$$\gamma(u) = \left(\frac{\cos(u)}{1+\sin^2(u)}, \frac{\sin(2u)}{2+2\sin^2(u)}\right).$$

Then $\gamma(u)$ *will evolve self-similarly under the CDF.*

Proof. Given the definition of $\gamma(u)$, we calculate its curvature to be:

$$\kappa(u) = \frac{3\cos(u)}{(1+sin^2(u))^{\frac{1}{2}}}.$$

We can thus find the arc-lenfth derivatives to be:

$$\kappa_s(u) = \frac{-6\sin(u)}{1+\sin^2(u)},$$

$$\kappa_{ss}(u) = \frac{-6\cos(u)(1-\sin^2(u))}{(1+\sin^2(u))^{\frac{3}{2}}}.$$

Furthermore, the unit normal can be found to be:

$$\nu(u) = \left(\frac{1-3\sin^2(u)}{(1+\sin^2(u))^{\frac{3}{2}}}, \frac{\sin(u)(2+\cos^2(u))}{(1+\sin^2(u))^{\frac{3}{2}}}\right).$$

Taking the inner product with $\gamma(u)$ yields:

$$\begin{aligned}\langle\gamma(u), \nu(u)\rangle &= \frac{\cos(u)(1-\sin^2(u))}{(1+\sin^2(u))^{\frac{3}{2}}} \\ &= \frac{-1}{6}\kappa_{ss}(u).\end{aligned}$$

Thus, by Lemma 2.3, $\gamma(u)$ is a self-similar solution to the CDF. □

Following the previous proposition, as we now know that the previously defined curve will flow self-similarly under CDF, we can find an explicit solution to the flow with this curve.

Corollary 2.6. *The curve $\gamma(u)$ defined in Proposition 2.5 will evolve under the CDF by:*

$$\sigma(u,t) = (12 - 12t)^{\frac{1}{2}}\gamma(u)$$

for $t \in [0,1]$.

Proof. Given that:

$$\langle \gamma, \nu \rangle = \frac{-1}{6}(\kappa_\gamma))_{ss},$$

$$\frac{\partial}{\partial t}\sigma = (\kappa_\sigma)_{ss}\nu,$$

and we can write $\sigma(u,t)$ as:

$$\sigma(u,t) = f(t)\gamma(u);$$

we need to find $f(t)$ such that:

$$f'(t)\gamma(u) = (\kappa_\sigma)_{ss}\nu.$$

We begin by calculating:

$$\begin{aligned}
\kappa_\sigma &= \frac{|\sigma' \times \sigma''|}{|\sigma'|^3} \\
&= \frac{|f(t)\gamma'(u) \times f(t)\gamma''(u)|}{f(t)\gamma'(u)|^3} \\
&= \frac{|f(t)|^2}{|f(t)|^3}\frac{\gamma' \times \gamma''|}{|\gamma'|^3} \\
&= \frac{1}{f(t)}\kappa_\gamma.
\end{aligned}$$

Thus we have:

$$(\kappa_\sigma)_{ss} = \frac{1}{f(t)}(\kappa_\gamma)_{ss}.$$

Now using the above information, we can find $f(t)$ as follows:

$$\begin{aligned} f'(t)\gamma(u) &= (\kappa_\sigma)_{ss}\nu \\ \implies f'(t)\langle\gamma,\nu\rangle &= (\kappa_\sigma)_{ss} \\ \implies f'(t)\langle\gamma,\nu\rangle &= \frac{1}{f(t)}(\kappa_\gamma)_{ss} \\ \implies f'(t)f(t) &= \frac{(\kappa_\gamma)_{ss}}{\langle\gamma,\nu\rangle} = -6 \end{aligned}$$

Thus, we have:

$$\frac{d}{dt}\left(\frac{1}{2}(f(t))^2\right) = -6.$$

We integrate to find:

$$f(t)^2 = -12t + f(0)^2 \implies f(t) = \sqrt{f(0)^2 - 12t}.$$

Furthermore, we have:

$$f(t) = 0 \iff t = \frac{1}{12}f(0)^2.$$

Which gives us our result:

$$\sigma(u,t) = (f(0)^2 - 12t)^{\frac{1}{2}}\gamma(u) \quad \text{for } t \in [0, \frac{1}{12}f(0)^2],$$

where we simply take $f(0)^2 = 12$.

□

3. Further Work

Where do we go from here? Ultimately we would like to find more self-similar solutions to the CDF. Generating these is tricky, however there are some techniques that can be combined with clever tactics to help us get there.

Progress at the moment has come to generating an ODE by use of Lemma 2.3 and some guesswork as to what other solutions may look like. One can easily obtain an ODE in the most general sense, that would find $\gamma_1(u)$ and $\gamma_2(u)$ such that (γ_1, γ_2) is a self-similar solution to the CDF. However, finding the functions which solve this ODE can be difficult. Related is to look at the ODE generated by trying to find a solution which is a graph over the circle, that is, a curve of the form $(r(u)\cos(u), r(u)\sin(u))$; this approach seems promising, but still needs some refinement. Perhaps some generalisation of this with controlled parameters will allow us to find some solutions.

The purpose in doing this is to find families of self-similar solutions. This will give us some idea of what kind of self-similar solutions exist. Similar evolution equations (see

Curve Shortening Flow) have been fully classified, that is, every kind of self-similar solution has been identified and formulas for families of self-similar solutions exist. Some heavy analysis could lead to a similar result for CDF.

4. Thanks

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