Topological Quantum Field Theory and Information Theory

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1 Introduction

In quantum field theory, the interactions between particles can be represented with a Feynman diagram (Figure 1). To eliminate the inner vertices, the whole diagram can be smoothed out to something along the lines of Figure 2. The boundaries at either end represent the particles, or vector spaces, and the interactions between them are represented by the object.

A Topological Quantum Field Theory (TQFT) takes some axioms from this physical background and throws away extraneous information to form a mathematical theory. It associates certain topological objects (cobordisms) with vector spaces. A Sutured

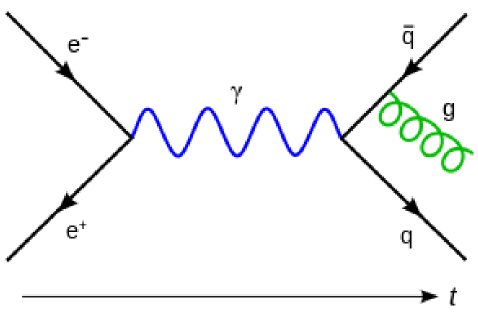

Figure 1: A Feynman diagram [6]

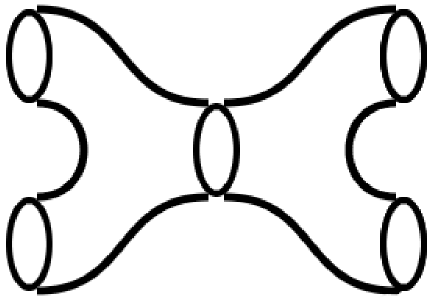

Figure 2: Smoothing out the Feynman diagram

Quadrangulated Field Theory (SQFT) is very similar, in that it associates other topological objects to vector spaces, but here the vector spaces are over the field $\mathbb{Z}_2$ with two basis vectors. These two basis vectors somehow suggest information, as in **1**'s and **0**'s.

There is a certain function, β, which was explored due to the possibility of a connection logic gates. Although nothing positive was discovered, the impossibility of logic gates in certain situations was proven. There were also other interesting facts about β that were explored.

2 Cobordisms

Definition 2.1. An n-dimensional manifold, M, is a topological space where every point has a neighbourhood homeomorphic to $\mathbb{R}^n$. Its boundary is denoted by δM.

Definition 2.2. A cobordism M between two manifolds (without boundary) of dimension n, Σ_1 and Σ_2, is a manifold of dimension $n+1$ such that $\delta M = \Sigma_1 \cup \Sigma_2$

Definition 2.3. An oriented cobordism is when both the manifold and its boundaries are oriented, in such a way as to have boundaries designated either *in* or *out*.

When we say cobordisms, we are actually refering to *equivalence classes* of cobordisms. Two cobordisms are in the same equivalence class if they are homeomorphic, that is, one can be deformed into another.

In this project we are working with surfaces, otherwise known as 2 dimensional cobordisms.

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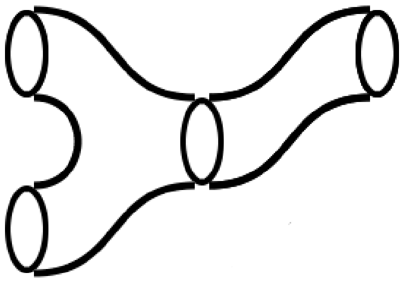

Figure 3: Joining together two cobordisms

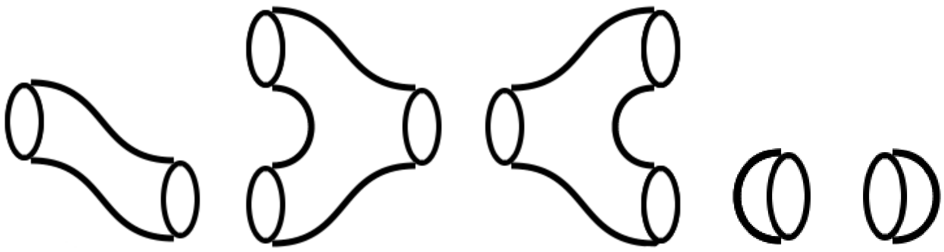

Figure 4: From left to right, the cobordisms known as identity, multiplication, comultiplication, unit and counit

2.1 Classification of Cobordisms

Two cobordisms, A and B, can be joined together by mapping some of the out boundaries of A to some of the in boundaries of B, as in Figure 3. As it turns out, all connected cobordisms can be made from the set of cobordisms given in Figure 4.

Theorem 2.4. *Connected cobordisms can be completely described by their genus, number of in boundaries, and number of out boundaries.*

Proof. By the Classification Theorem of Surfaces [5], all orientable surfaces can be described completely by their genus and number of boundaries. Oriented cobordisms are complicated by their use of in and out boundaries, but by differentiating these we are able to arrive at a complete classification. □

Theorem 2.5. *Any connected cobordism can be constructed from the composition of the set of cobordisms given in Figure 4*

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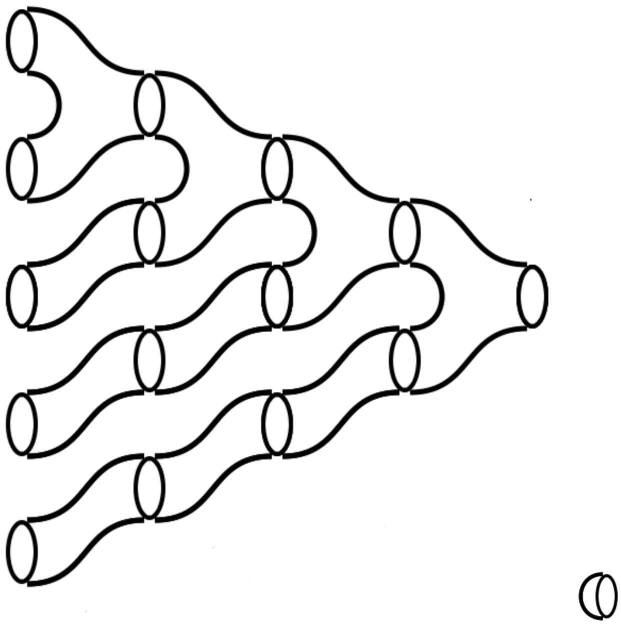

Figure 5: Two cobordisms, one with 5 in boundaries and one with 0.

Proof. Consider a cobordism, M, with n in boundaries, m out boundaries, and genus g. We will begin by constructing the pieces we need.

Firstly, add multiplication cobordisms together to form a piece with n in boundaries and one out boundary, as shown in Figure 5. Caps are required for $n = 0$.

Use the same idea with comultiplication to form a piece with m out boundaries and one in boundary (Figure 6). Now put together two pants as so, to form a torus with two discs removed (Figure 7). Attaching g of these together will form a surface with genus g (Figure 8) Now we can construct M by attaching each of these pieces together (Figure 9).

□

3 Topological Quantum Field Theory

3.1 Category Theory

A category is a set of objects which have morphisms, or arrows, going between them. Every object has an identity morphism, which is the morphism from an object to itself.

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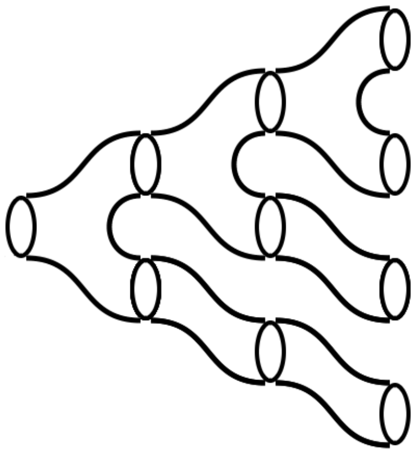


Figure 6: A cobordism with 4 out boundaries.

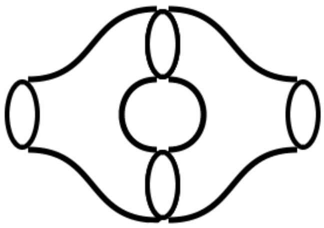


Figure 7: A cobordism of genus 1.

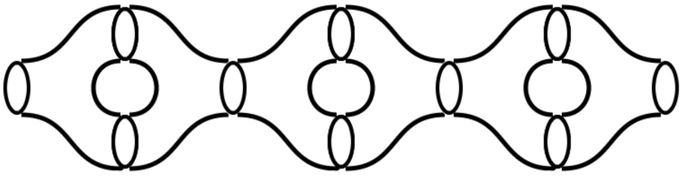


Figure 8: A cobordism of genus 3.

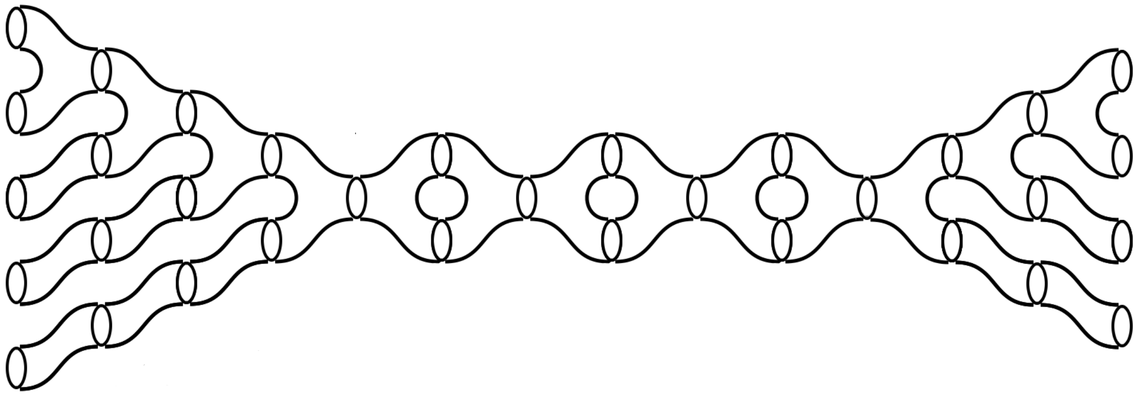


Figure 9: A cobordism of genus 3, with 5 in boundaries and 4 out boundaries.

There exists a binary operation, composition of morphisms, which is associative and obeys $f \circ 1 = 1 \circ f = f$, where f is a morphism and 1 is the identity morphism mentioned earlier. Some examples include:

- Sets, where the morphisms are functions from one set to another
- Groups, with homomorphisms being the morphisms
- Vector spaces under the field k, with linear transformations as morphisms

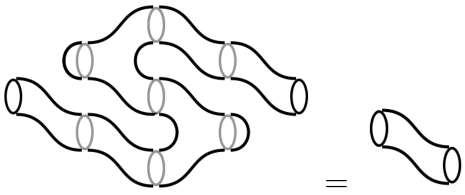
The category of cobordisms is known as **ncob**.

3.2 Topological Quantum Field Theory

Consider the category of Vector spaces, $\mathbf{Vect_k}$, over the ground field k and maps between them, for example $m : V \otimes V \mapsto k$, where V is a vector space.

A Topological Quantum Field Theory, A, is a map from $\mathbf{Vect_k}$ to **nCob** satisfies the following axioms:

1. Topologically equivalent cobordisms, those in the same class, represent the same map: $M_1 \cong M_2 \implies A(M_1) = A(M_2)$



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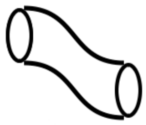
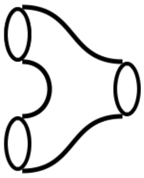
2. The identity map is the cylinder



3. Composition of cobordisms goes to composition of linear maps: $M_1 \circ M_2 = M_3 \implies A(M_1) \circ A(M_2) = A(M_3)$

4. Disjoint union goes to tensor product, eg $V \otimes V \mapsto V$



5. The empty boundary goes to the ground field k, eg $k \mapsto V$

In category theory, numbers 1, 2 and 3 are equivalent to being a functor, and 4 and 5 mean monoidal. There is another axiom equivalent to begin symmetric. [2] Thus, a category theoretic definition of a TQFT is:

Definition 3.1. A TQFT is a symmetric, monoidal functor from the category of (n-dimensional) cobordisms, **ncob**, to vector spaces $\mathbf{Vect_k}$.

4 Sutures and Information Theory

Instead of cobordisms, we now work with *occupied surfaces*.

Definition 4.1. An occupied surface is an orientable surface with boundary, with signed vertices on the boundary, alternating between positive and negative. [1]

On an occupied surface can be drawn *sutures*, lines which divide the surface into positive and negative regions.

Definition 4.2. To quadrangulate a surface, we draw lines from vertex to vertex, dividing the surface into squares. Lines are drawn between vertices of opposite sign.

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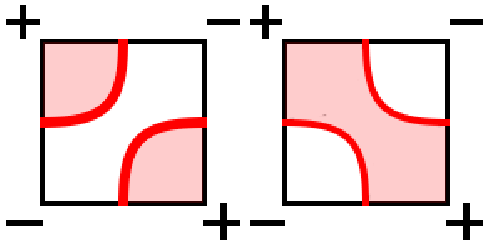

Figure 10: Two simple sets of sutures for the square

The Euler class of a sutured, occupied surface is $\sum e_+ - e_-$, that is, the *Euler characteristic* of the positive regions minus the Euler characteristic of the negative regions.

Definition 4.3. Euler characteristic of a connected oriented surface of genus g with n boundary components is given by:

$$e = 2 - 2g - n$$

Consider an occupied disc with four vertices, a square. There are two possible suturings for it, excluding those with closed curves, shown in Figure 10.

These two squares have Euler class 1 and -1, respectively, and will be known as **1**'s and **0**. The names given are for to suggest links to information theory. The vector space is over the field $\mathbb{Z}_2$, which has elements 0 and 1. To somewhat mitigate confusion, these two squares will be given in bold while elements of the field $\mathbb{Z}_2$ will not be.

Definition 4.4. A quadrangulation is known as basic if each cell in the quadrangulation is either a **1** or a **0**.

Theorem 4.5. *Every occupied surface has a basic quadrangulation, except in the case of a disc with two vertices. [1]*

4.1 Sutured Quadrangulated Field Theory

Consider a vector space with basis **1** and **0** over the field $\mathbf{Z}_2$. Let us call it V. This vector space can be associated to an occupied surface in a similar way to how a TQFT associates vector spaces with cobordisms.

Consider a quadrangulated occupied surface. A Sutured Quadrangulated Field Theory associates this to the vector space $V^{\otimes n}$, where n is the number of squares in the quadrangulation of that occupied surface.

1 and **0** correspond to the two basis sutures from Figure 10. Elements in this vector space $V^{\otimes n}$ correspond to different suturings of the surface.

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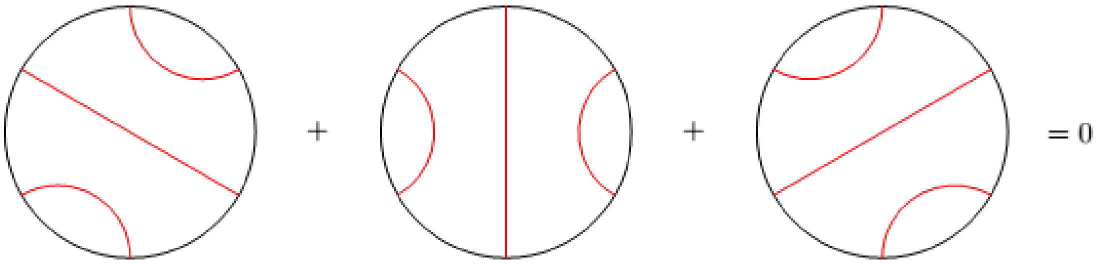

Figure 11: The Bypass relation [1]

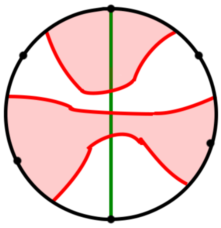

Figure 12: A sutured, quadrangulated disc, which is not a basic element of the vector space.

4.2 Bypass Relation

Pictorially, the Bypass Relation looks like Figure 11.

There is a theorem in SQFT, where the above discs correspond to elements in the vector space which sum to zero. It is also a method for manipulating the sutures in order to represent a particular non-basic suturing as a sum of basic sutures, for example in Figure 12.

What element of the vector space is represented by such a suturing? We can isolate

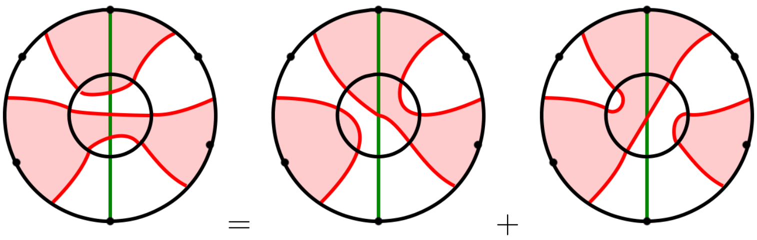

Figure 13: Using the Bypass relation to solve the previous problem.

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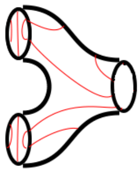

Figure 14: Two discs from the bypass relation forming a new disc

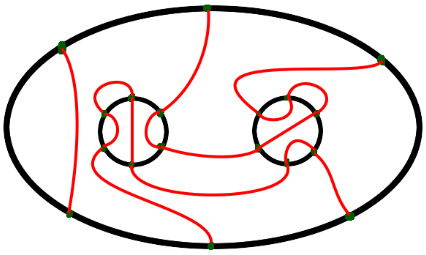

Figure 15: A function which takes two elements from the bypass relation to something which is neither

part of the disc, then use the bypass relation to represent this element as a sum of two other elements, as in Figure 13. We can see that our original disc is the element $\mathbf{0} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{0}$.

4.3 Logic Gates

The Bypass relation presents three types of discs, which could potentially be used as the values True, False, and some sort of Indeterminate. The idea was to form a logic gate from something similar to a cobordism (Figure 14). This can also be represented as a disc with two holes (15). The sutures in this example do not form a logic gate. As it turns out, it is impossible.

Theorem 4.6. *Let the elements of the bypass relation denote the values True, False, and Indeterminate. Let there be an occupied disc with one hole, and six vertices to each boundary. Then there is no set of sutures such that a value can be entered into the hole to give another value to the disc as a whole, giving a NOT gate.*

Proof. Consider a suture originating on the inner boundaries. It can either end on the same boundary or the outer boundary. If it ends on the same boundary, there will be

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at least one value for which this forms a closed loop. None of the values have a closed loop in them, so the suture cannot form closed loops.

Thus, all the sutures must go from the inner boundary to the outer boundary. However, up to rotation, there is only one way to do this and it does not form a NOT gate. □

Theorem 4.7. *Let the elements of the bypass relation denote the values True, False, and Indeterminate. Let D be an occupied disc with two holes, and six vertices to each boundary. Then there is no set of sutures such that two values can be entered into the two holes to give a value.*

Proof. None of the values have a closed loop in them, so the suture cannot form closed loops. There are 6 possible slots in a boundary that sutures can begin and end on. A suture which begins and ends on the same boundary takes up two slots, but one which ends on a different boundary takes up only one. Thus there must be an even number of sutures from any particular boundary ending on another.

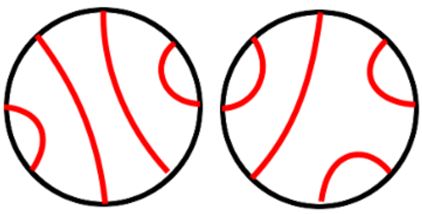
Consider a suture originating in one of the inner boundaries. It can either end on the same boundary, the outer boundary or the other inner boundary.

If it ends on the same boundary, there is the same argument as in Theorem 4.6. Thus let us assume there are no sutures from an inner boundary to itself.

If it ends on the other inner boundary, in order to avoid sutures from an inner boundary to itself, all the sutures from one inner boundary to the other must be in a row. The other sutures must go to the outer boundary. Thus, there are only 4 possibilities, none of which is a logic gate. □

5 Cylinders and β

Consider two sutured discs, Γ_1 and Γ_2, with the same number of vertices on the boundary.



Place them at either end of a cylinder, face up, as shown in Figure 16. Now draw lines on the sides, rotating by half each time in order to preserve sign. This describes

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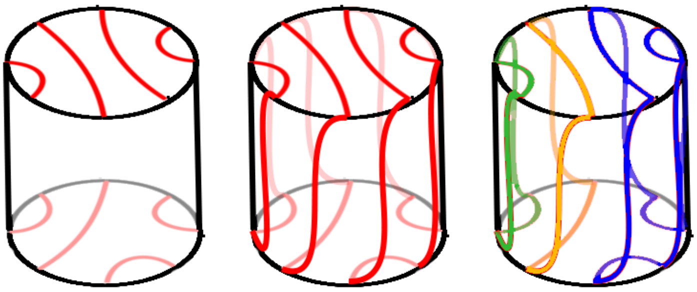

Figure 16: The function β

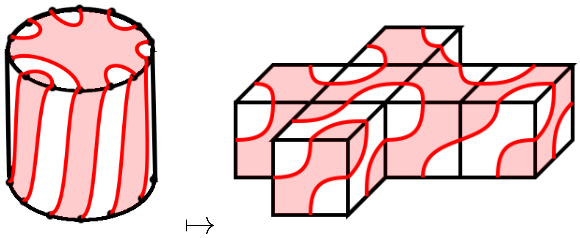

Figure 17: Turning the cylinder into cubes.

a function, $\beta(\Gamma_1, \Gamma_2)$. If the disc has corresponding vector space $V^{\otimes n}$ then $\beta : V^{\otimes n} \otimes V^{\otimes n} \mapsto \mathbb{Z}_2$. If the sutures are now homeomorphic to a single closed loop, $\beta(\Gamma_1, \Gamma_2) = 1$. If there are multiple loops, $\beta(\Gamma_1, \Gamma_2) = 0$. Remember, the lack of bold here indicates the numbers 0 and 1, not the elements of the vector space.

In this particular example, it is clear that there are multiple loops, so $\beta(\Gamma_1, \Gamma_2) = 0$.

6 Cubes

In an SQFT, an occupied surface can be represented as an element of the vector space V. Instead of taking two discs and finding the value β takes by drawing sutures and tracing around them, can we take two elements of the vector space V and determine what β is?

One way of looking at this problem is to quadrangulate each disc into squares (Figure 17). Then look at each resulting cube and find the corresponding β (Figure 18). Then glue the cubes together, knowing how this affects the value of β for the whole.

It would be nice if, given any two Γ_1 and Γ_2, a particular quadrangulation could

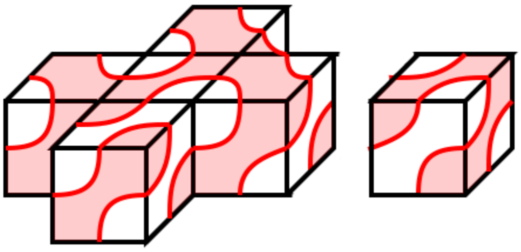

Figure 18: Dividing into two groups.

be found which is simultaneously basic in both; however, a counterexample is shown in Figure 19.

It is clear that there are only two distinct spots from which to begin an arc of quadrangulation. In order for a quadrangulation to be basic, each arc may intersect a suture of each colour only once. It is clear from the image that there is no possible way to do this.

However, if we allow adding inner vertices to which the arcs of quadrangulation can be anchored, we arrive at something known as a slack quadrangulation. It is clear that there can always be a slack quadrangulation simultaneously basic in two discs, but is there an algorithm by which we can find a slack quadrangulation with a small number of slack vertices? Given a quadrangulation which is basic in both discs, can we find β?

A useful result which makes progress towards answering these questions is given below.

Lemma 6.1. *Let D be an occupied disc with quadrangulation, Q, and two sets of sutures, Γ_1 and Γ_2. Suppose D decomposes into two discs along an arc of Q, such that $D = D' \sqcup \amalg D''$. Now we have sutures:*

$\Gamma_1' = \Gamma_1 \cap D'$
$\Gamma_2' = \Gamma_2 \cap D'$
$\Gamma_1'' = \Gamma_1 \cap D''$
$\Gamma_2'' = \Gamma_2 \cap D''$

If $\beta(\Gamma_1', \Gamma_2') = 1$ *and* $\beta(\Gamma_1'', \Gamma_2'') = 1$*, then* $\beta(\Gamma_1, \Gamma_2) = 1$

Proof. The face at which the left surface is to be glued is shown on the left in Figure 20. There is only one possible way for the sutures to be drawn that results in the surface having only a single loop; that is, $\beta = 1$. Similarly, there is only one way to draw the shape on the right. When joined together, it is clear that there will be only

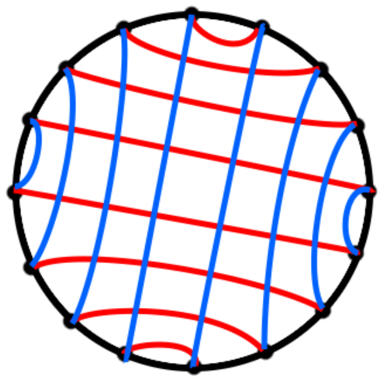

Figure 19: A disc which cannot be quadrangulated in such a way that the two sets of sutures are simultaneously basic. Each set of sutures on the disc is represented with a different colour.

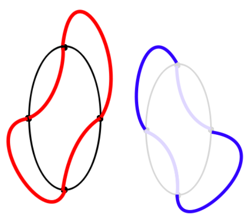

Figure 20: This is a stylised drawing of two cubes, with only the faces to be joined seen concretely; the other sides are not shown except for their sutures. The second face is shown from behind.

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Figure 21: This is a stylised drawing of two cubes, with only the faces to be joined seen concretely; the other sides are not shown except for their sutures. The second face is shown from behind.

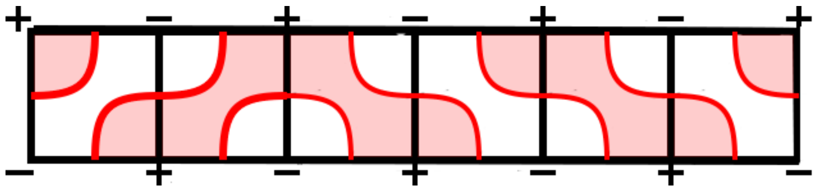

Figure 22: A disc given by the element $\mathbf{1} \otimes \mathbf{0} \otimes \mathbf{0} \otimes \mathbf{1} \otimes \mathbf{0} \otimes \mathbf{1} \otimes$

a single loop (Figure 21). □

Given a *linear quadrangulation*, one can easily find β from the string of digits. A linear quadrangulation is one where the disc is given by a string of $\mathbf{1}$'s and $\mathbf{0}$'s arranged in a line, as shown in Figure 22.

Theorem 6.2. *Given a linear quadrangulation and two elements of the corresponding vector space,* $\mathbf{a} = a_1 \otimes a_2 \otimes \ldots \otimes a_n$ *and* $\mathbf{b} = b_1 \otimes b_2 \otimes \ldots \otimes b_n$*,* $\beta(\mathbf{a}, \mathbf{b})$ *can be found from the following equation:*

$$\prod_{i=0}^{n} \left[(-1)^i (a_i - b_i) + 1 \right]$$

If the result is a 1*,* $\beta(\mathbf{a}, \mathbf{b}) = 1$ *If not,* $\beta(\mathbf{a}, \mathbf{b}) = 0$

Proof. Decompose the discs into their constituent $\mathbf{1}$'s and $\mathbf{0}$'s, so that now instead of composing disc Γ_1 with disc Γ_2, you are now composing some string of $\mathbf{1}$'s and $\mathbf{0}$'s with

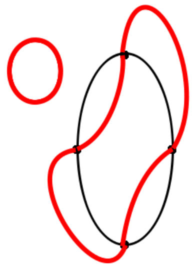

Figure 23: First type of zero cube, drawn with only sutures and face to be glued

another string of **1**'s and **0**'s. Now instead of a cylinder there is a string of cubes, all joined together.

Remembering the order of these cubes, we now break apart the string of cubes and consider each cube separately. We know that $\beta(\mathbf{1},\mathbf{1}) = 1$, $\beta(\mathbf{0},\mathbf{1}) = 0$, $\beta(\mathbf{1},\mathbf{0}) = 0$, and $\beta(\mathbf{0},\mathbf{0}) = 1$.

By Lemma 6.1, if there is a sequence of cubes glued together for which $\beta = 1$, and some other sequence for which $\beta = 1$, then glueing these two sequences together will result in a sequence for which $\beta = 1$. (in fact, adding a $\beta = 1$ cube to anything does not change it)

There are two types of cubes for which $\beta = 0$, that of Figure 23 and that of Figure 24.

With the first, it can be seen that glueing this to any sequence will add an extra loop and so $\beta = 0$. The second type can be glued to itself to result in a sequence for which $\beta = 1$. This cube, or in fact any combination of this cube and the $\beta = 1$ type, the surface discussed above, can only look like Figure 24. When glued, the sutures will form only one loop, as in Figure 25. Thus, if there are an odd number of the second type there will be an extra loop and so β will be zero.

The first type of cube appears when there is a **1** above a **0** in an odd position, or when there is a **0** above a **1** in an even position. The second occurs elsewhere.

Combining these ideas, an algorithm for discovering whether two given sequences of numbers results in a cylinder for which $\beta = 1$ can be given:

1. List two sequences of **1**'s and **0**'s.

2. Pair the first number with the first, second with the second, etc.

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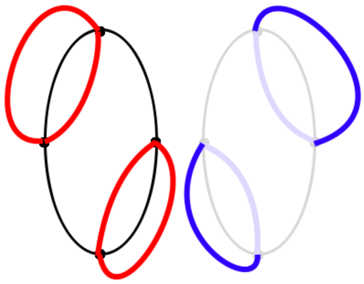

Figure 24: Two type B zero cubes, side by side

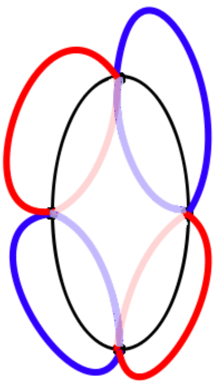

Figure 25: Two type B zero cubes composed to form an object for which $\beta = 0$

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3. If the two numbers are the same, write down a 0. If $(\mathbf{1}, \mathbf{0})$ write down a 1, and if $(\mathbf{0}, \mathbf{1})$ write down a -1.
4. Multiply each number alternately by $+1$ and -1. That's why previously $(\mathbf{1}, \mathbf{1})$ and $(\mathbf{0}, \mathbf{0})$ were given the value of 0, so that they were not changed by this.
5. Add $1 \mod 3$ to each number.
6. Multiply together. Multiplying by 1 does not change anything, just as gluing a $(1, 1)$ or a $(0, 0)$ does not change anything. Multiplying by -1 functions just as glueing the second type of zero cube does, and multplying by 0 instantly gives the whole thing a value of zero, like the first type of zero cube.
7. If the result was 1, $\beta = 1$, otherwise $\beta = 0$

These steps are equivalent to the formula given in the theorem. □

7 Conclusion

In this project the idea of a TQFT and its relation to information theory was explored. Topology, in particular surfaces, and category theory were introduced as background for the SQFT discussed here. The exploration of β led to the results given here about decomposition of cubes.

One area for further research is whether a logic gate could be found somewhere; whether there is some sort of topological representation of logic. It also would have been ideal to further generalise the decomposition into cubes, perhaps link it to quantum groups or find a matrix representation, however due to time constraints this was not possible.

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