EHRENFEUCHT-FRAÏSSÉ GAMES IN FINITE ALGEBRAS

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Abstract. Ehrenfeucht-Fraïssé games give us a nice tool for proving inexpressibility results in first-order logic over finite structures. Typically, the games are restricted to a relational setting. But what if we are interested in playing Ehrenfeucht-Fraïssé games over algebraic structures? We could play the games on graphs of algebras; however the formulation of algebraic structures as relational structures is not particularly natural from an algebraic perspective. In this paper, we look at a reformulation of Ehrenfeucht-Fraïssé games that can be played over structures where both relations and operations are allowed. In fact, the games properly generalise the standard formulation, though adjusting the standard relational structure proofs to this new setting is an intricate process.

1. Preliminaries.

We begin with some standard definitions from first-order (FO) logic.

Definition 1.1. A *vocabulary* (or *signature*) σ is a collection of constant symbols (denoted $c_1, c_2, \ldots, c_m, \ldots$), relation symbols $(r_1, r_2, \ldots, r_m, \ldots)$ and function symbols $(f_1, f_2, \ldots, f_m, \ldots)$. Each relation and function symbol has an associated arity k.

A σ*-structure* (also called a *model*) $\mathbf{A} = \langle A; \mathcal{C} = \{c_i^A\}, \mathcal{R} = \{r_i^A\}, \mathcal{F} = \{f_i^A\}\rangle$ consists of a universe $A \neq \varnothing$ together with an interpretation of

- each constant symbol c_i from σ as a nullary operation;
- each k-ary relation symbol r_i from σ as a k-ary relation on A; and
- each k-ary function symbol f_i from σ as an operation $f_i^A : A^n \to A$.

If $\mathcal{R} = \varnothing$, then $\mathbf{A}$ is an algebra; if $\mathcal{F} = \varnothing$ then $\mathbf{A}$ is a relational structure. We denote algebraic signatures by $\mathcal{F}$ and relational signatures by $\mathcal{R}$. A structure $\mathbf{A}$ is called finite if its universe A is a finite set. Note that "type" and "language" are often used as a synonyms for "signature".

Definition 1.2. We inductively define *terms* and *formulæ* of FO logic by rules (1)–(3) and (4)–(7) respectively.

(1) **Variables.** Each variable x is a term.
(2) **Constants.** Each constant symbol c is a term.
(3) **Functions.** If $t_1, \ldots, t_k$ are terms and f is a k-ary function symbol, then $f(t_1, \ldots, t_k)$ is a term.
(4) **Equality.** If t_1 and t_2 are terms, then $t_1 \approx t_2$ is an atomic formula.
(5) **Relations.** If $t_1, \ldots, t_k$ are terms and r is a k-ary relation symbol, then $r(t_1, \ldots, t_k)$ is an atomic formula.

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(6) **Binary Connectives and Negation.** If φ_1, φ_2 are formulæ then $\varphi_1 \wedge \varphi_2, \varphi_1 \vee \varphi_2$ and $\neg\varphi$ are formulæ.
(7) **Quantifiers.** If φ is a formula, then $\exists x\varphi$ and $\forall x\varphi$ are formulæ.

Definition 1.3. In a formula or a term, a variable is either *free* or *bound*. Intuitively, a variable is free if it is not quantified. Free variables of a formula or a term are defined as follows.

- The only free variable of a term x is x; a constant term c does not have free variables.
- Free variables of $t_1 \approx t_2$ are the free variables of t_1 and t_2; free variables of $r(t_1, \ldots, t_k)$ or $f(t_1, \ldots, t_n)$ are the free variables of $t_1, \ldots, t_k$.
- Negation does not change the list of free variables; the free variables of $\varphi_1 \wedge \varphi_2$ and $\varphi_1 \vee \varphi_2$ are the free variables of φ_1 and φ_2.
- Free variables of $\exists x\varphi$ and $\forall x\varphi$ are the free variables of φ except x.

Variables that are not free are called bound. A *sentence* is a formula with out free variables.

Finally we introduce the notion of satisfaction $\mathbf{A} \models \Phi$ and some associated notation.

- If $\vec{x}$ is a tuple of all the free variables of φ, we write $\varphi(\vec{x})$.
- If $\mathbf{A}$ is a structure of some type, Φ is a sentence in the language of $\mathbf{A}$, and Φ is true in $\mathbf{A}$, we write $\mathbf{A} \models \Phi$.
- Given a set S of FO sentences, we say that two structures, $\mathbf{A}$ and $\mathbf{B}$, of the same type, *agree on* S if for every sentence Φ of S it is the case that $\mathbf{A} \models \Phi$ iff $\mathbf{B} \models \Phi$.
- If $\Psi(\vec{x}) \equiv \exists y\varphi(y, \vec{x})$, then $\mathbf{A} \models \varphi(\vec{a})$ iff $\mathbf{A} \models \varphi(\acute{a}, \vec{a})$ for some $\acute{a} \in A$.

2. Ehrenfeucht-Fraïssé games

Ehrenfeucht-Fraïssé games are a theoretical tool for evaluating the logical similarities between two mathematical structures. Instead of working with individual finite structures, one works with an indexed family of pairs of finite structures (of increasing size). The games can be used in this setting to show that properties of finite structures cannot be defined in first order logic.

As an example, Ehrenfeucht-Fraïssé games can be used to show that a linear order of length 2^k (k being the number of rounds played of the game) is indistinguishable from one that has length greater than 2^k and, in turn, this can be used to show that the property of having even cardinality is not first order definable over linear orders. In a similar way, the games can be used to show that connectedness, being a tree, and being a path are not first order properties of a graph.

Indeed, Ehrenfeucht-Fraïssé games provide a complete methodology for proving inexpressibility results. In this role, the games have found particular utility at the finite level, even though finite structures are logically equivalent (in first order logic) if and only if they are isomorphic. Standard model theoretic techniques for proving inexpressibility results, such as the Compactness Theorem or the Löwenheim-Skolem Theorem, are not generally applicable at the finite level.

In the Ehrenfeucht-Fraïssé game, there are two players called Spoiler and Duplicator. The board of the game consists two relational structures (or $\mathcal{R}$-structures)

A and **B**. The goal of Spoiler is to show the two structures as distinct; the goal of Duplicator is to show they are the not different.

The players play a certain number of rounds. Each round consists of the following rules.

(1) Spoiler picks a structure; either **A** or **B**.
(2) Spoiler makes a move by selecting an element of that structure; either $a \in \mathbf{A}$ or $b \in \mathbf{B}$.
(3) Duplicator responds by selecting an element in the other structure.

After n rounds of an Ehrenfeucht-Fraïssé game, we have moves $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$. Let $c_1, \ldots, c_l$ be the constant symbols in $\mathcal{R}$; then $\vec{c}^A$ denotes $(c_1^A, \ldots, c_l^A)$ and likewise for $\vec{c}^B$. Then Duplicator wins the n-round game played on **A** and **B** provided the map that sends each a_i into b_i and each ${c_j}^A$ into ${c_j}^B$ is an isomorphism between the substructures of **A** and **B** generated by $\{a_1, \ldots, a_n, \vec{c}^A\}$ and $\{b_1, \ldots, b_n, \vec{c}^B\}$ respectively. Intuitively speaking, Duplicator wins the n-round game if he/she can duplicate any of the n moves that Spoiler makes.
If Duplicator can win the n-round game regardless of how Spoiler plays, we say that Duplicator has an n-round winning strategy and we write $\mathbf{A} \equiv_n \mathbf{B}$.

3. Some Concepts.

3.1. **The graph of an algebra.** Given any algebra (or even any partial algebra) $\mathbf{A} = \langle A; f_1, \ldots, f_m \rangle$, one can define a relational structure denoted by graph($\mathbf{A}$) by replacing each operation f_i by a relation r_i defined by

$$r_i := \{(a_1, \ldots, a_k, a_{k+1}) \mid f_i(a_1, \ldots, a_k) = a_{k+1}\}.$$

Note that the arity of the relation r_i is one greater than the arity of the operation it replaces.

Lemma 3.1. *The following are equivalent for a class K of finite algebras (of the same type):*

(1) there is a first order sentence Φ such that K is the class of all finite algebras satisfying Φ.
(2) there is a first order sentence Ψ such that the class

$$\{\mathrm{graph}(\mathbf{A}) \mid A \in K\}$$

is the class of all relational structures satisfying Ψ.

Proof. (Sketch) Let **A** be some algebra in the class K and let f be some operation in the signature of **A**.
((2) $\implies$ (1)). In Φ, replace $(x_1, \ldots, x_n, x_0) \in \mathrm{graph}(f)$ by $f(x_1, \ldots, x_n) \approx x_0$.
((1) $\implies$ (2)). The atomic formulae (from which sentences are built) are equalities between terms. Replace these within Ψ by formulae involving the graphs of the operations. The details use induction, but the idea is straightforward. For example, if the signature contains a binary operation $*$ then the equation $x*(y*z) = (x*y)*z$

becomes

$$(\exists w_1, w_2, w_3)\big((y, z, w_1) \in \text{graph}(*) \ \ \& \ \ (x, y, w_2) \in \text{graph}(*) \\ \& \ \ (x, w_1, w_3) \in \text{graph}(*) \ \ \& \ \ (z, w_2, w_3) \in \text{graph}(*)\big).$$

Finally, we must take the conjunction of this sentence obtained by the above with a further statement asserting that the relations corresponding to operations are actually the graphs of operations. Using the binary $*$ case again, we could use the following:

$$\forall x \forall y \exists z \forall z_1 \, (x, y, z) \in \text{graph}(*) \ \ \& \ \ (x, y, z_1) \in \text{graph} \Rightarrow z \approx z_1.$$

□

4. Playing Ehrenfeucht-Fraïssé games on algebras

Standard presentations of Ehrenfeucht-Fraïssé games (like the one given above) make a point of restricting to relational structures (no operations, just relations and constants). But what if we want to play games on algebraic structures? Lemma 3.1 shows that conventional Ehrenfeucht-Fraïssé games can be played on the graphs of algebras; however the formulation of algebraic structures as relational structures is not particularly natural from an algebraic perspective. Instead, we propose that the following game is more natural algebraically.

- **Our New Game.** Our variant of the Ehrenfeucht-Fraïssé game will have exactly the same rules as before, but interpreted in the algebra instead of its graph. Thus, after n rounds, the game will have generated some list $a_1, \dots, a_n$ and $b_1, \dots, b_n$. Duplicator wins provided that the map that sends each a_i into b_i and each ${c_j}^A$ into ${c_j}^B$ *extends* to an isomorphism from the subalgebra of $\mathbf{A}$ generated by $\{a_1, \dots, a_n, \vec{c}^A\}$ to the subalgebra of $\mathbf{B}$ generated by $\{b_1, \dots, b_n, \vec{c}^B\}$.

Observe that the algebraic formulation of the game properly generalises the standard one in that duplicator wins provided that the bijection created based on the players' selection extends to an isomorphism on the subalgebras generated by these elements. In a relational signature, there is no generating power beyond the selected elements and the constants, so the new game coincides with the original.

Now recall the definition of quantifier rank, this time interpreted in algebraic signatures (the definition is identical to the relational case):

- If ϕ is an atomic formula then $\text{qr}(\phi) = 0$;
- $\text{qr}^+(\phi_1 \vee \phi_2) = \text{qr}^+(\phi_1 \ \ \& \ \ \phi_2) = \max\{\text{qr}^+(\phi_1), \text{qr}^+(\phi_2)\}$;
- $\text{qr}^+(\neg\phi) = \text{qr}^+(\phi)$;
- $\text{qr}^+(\forall x \, \phi) = \text{qr}^+(\exists x \, \phi) = \text{qr}^+(\phi) + 1$.

The following theorem is an algebraic variant of the Ehrenfeucht-Fraïssé Theorem. We use the notation $\text{FO}[n]$ for all FO formulae of algebraic quantifier rank up to n.

Theorem 4.1. *The following are equivalent for two* finite *algebras* $\mathbf{A}$ *and* $\mathbf{B}$ *of the same type:*

(1) $\mathbf{A} \equiv_n \mathbf{B}$*; that is, Duplicator has a winning strategy in an* n*-round play of our new game.*

(2) $\mathbf{A}$ *and* $\mathbf{B}$ *agree on* FO$[n]$.

Similar to that of the Ehrenfeucht-Fraïssé Theorem, Theorem 4.1 gives rise to a general methodology for proving inexpressibility results, *which is complete for uniformly locally finite classes of finite algebras.*

Corollary 4.2. *A property $\mathcal{P}$ of is not expressible in FO, if for every $n \in \mathbb{N}$, there exist two* finite *algebras $\mathbf{A_n}$ and $\mathbf{B_n}$, such that:*

- $\mathbf{A_n} \equiv_n \mathbf{B_n}$.
- $\mathbf{A_n}$ *has property $\mathcal{P}$, and* $\mathbf{B}$ *does not.*

Compared to the standard formulations of Theorem 4.1 and Corollary 4.2, the proofs involve some extra intricacies. Most of the next section is devoted to handling these complications. We finish off by giving the proofs of the above results; which turn out to be relatively straightforward.

5. Proof Of Theorem 4.1 and Corollary 4.2

The notion of quantifier rank has an extra complication in the algebraic setting: even up to logical equivalence there may be infinitely many different quantifier rank 0 sentences. For example, if the signature contains a constant c and a binary operation $*$, then each of the following atomic sentences are different: $c \approx c * c$, $c \approx c * (c * c)$, $c \approx c * (c * (c * c))$, $\dots$. However, when playing Ehrenfeucht-Fraïssé games on finite algebras, there may occur some collapse of these sentences, and we utilise this in the proofs.

5.1. Terms and a Technical Lemma.

A *term* in some algebraic signature $\mathcal{F}$ is a well formed expression built from combining variables using the operation symbols in $\mathcal{F}$. (Recall that we consider constants as nullary operations. This in a signature $\{\cdot, c, d\}$ of type $\langle 2, 0, 0\rangle$, the expressions c and $(c \cdot (d \cdot c)) \cdot c$ are examples of terms in which there are no variables used.) We use notations such as $t(x_1, \dots, x_n)$ to denote terms whose variables are amongst $x_1, \dots, x_n$.

Define the *height* of a term as follows: variables are of height 0. If f is a fundamental operation of arity n and the maximum height amongst some set of terms $t_1, \dots, t_n$ is k, then $f(t_1, \dots, t_n)$ is of height $k + 1$. We interpret this as defining the height of nullaries to be 1.

Let $\mathbf{A}$ be an algebraic structure of some finite type. Each term $t(x_1, \dots, x_n)$ in the signature of $\mathbf{A}$ induces a function from A^n into A: simply evaluate t at a given tuple. We use $t^{\mathbf{A}}(x_1, \dots, x_n)$ to denote the function on $\mathbf{A}$ corresponding to t (recall that t itself is just an expression). A subtlety is that the variables $x_1, \dots, x_n$ do not have to all appear in the expression t (though all variables in t must be amongst $x_1, \dots, x_n$). Thus the arity n of the function $t^{\mathbf{A}}(x_1, \dots, x_n)$ depends on the variables we have specified.

Note that if $s \approx t$ is an equation between two terms s, t whose variables are amongst $x_1, \dots, x_n$, then $\mathbf{A} \models s \approx t$ if and only if $s^{\mathbf{A}}(x_1, \dots, x_n)$ and $t^{\mathbf{A}}(x_1, \dots, x_n)$ are identical functions.

The following lemmas are phrased in some fixed finite algebraic signature $\mathcal{F}$.

Lemma 5.1. *Fix a finite algebra* $\mathbf{A}$. *Let* $X = \{x_1, \dots, x_n\}$ *be a finite set of variables, and let* $T(X)$ *denote the set of all terms whose variables are amongst* X. *Define an equivalence relation* $\equiv_{\mathbf{A}}$ *on* $T(X)$ *by* $s(x_1, \dots, x_n) \equiv_{\mathbf{A}} t(x_1, \dots, x_n)$ *if* $s^{\mathbf{A}}(x_1, \dots, x_n) = t^{\mathbf{A}}(x_1, \dots, x_n)$. *Then* $\equiv_{\mathbf{A}}$ *is a congruence and* $T(X)/{\equiv_{\mathbf{A}}}$ *has at most* $|A|^{|A|^n}$ *equivalence classes.*

Proof. First, $\equiv_{\mathbf{A}}$ has at most $|A|^{|A|^n}$ classes because this is the number of functions from A^n into A (and not all of these need come from terms). Now for the congruence claim. Let $f \in \mathcal{F}$ be k-ary and consider terms $s_1, \dots, s_k$ and $t_1, \dots, t_k$ from $T(X)$ with $s_i \equiv_A t_i$ for $i = 1, \dots, k$. Then for each i we have $s_i^{\mathbf{A}} = t_i^{\mathbf{A}}$, so that $f^{\mathbf{A}}(s_1^{\mathbf{A}}, \dots, s_k^{\mathbf{A}}) = f^{\mathbf{A}}(t_1^{\mathbf{A}}, \dots, t_k^{\mathbf{A}})$, which implies that $f(s_1, \dots, s_k) \equiv_{\mathbf{A}} f(t_1, \dots, t_k)$. That is, $\equiv_{\mathbf{A}}$ is a congruence. □

The following lemma is an algebraic variant of Lemma 3.13 of Libkin [2].

Lemma 5.2. *Let* k, m, n *be natural numbers.*

(1) *Every atomic subformula of a rank* k *formula in* m *free variables* $x_0, \dots, x_{m-1}$ *involves at most* $m + k$ *variables.*
(2) *Up to logical equivalence there are only finitely many distinct rank* k *formulæ in the* m *free variables* $x_0, \dots, x_{m-1}$ *whose terms are of height at most* n.

Proof. Both statements are proved by induction over k (the number m varies: for each k we ask the statement be true of any m). The first is trivial in the $k = 0$ case. Assume now that for any m, an atomic subformula of a rank k formula in m free variables $x_0, \dots, x_{m-1}$ involves at most $m + k$ variables. If $\Phi(x_0, \dots, x_{m-1})$ is of rank $k + 1$, it is a Boolean combination of formulæ of the form $\exists x_m \Psi(x_0, \dots, x_{m-1}, x_m)$, where $\Psi(x_0, \dots, x_{m-1}, x_m)$ is of rank k. All atomic subformulæ of Φ appear in these rank k subformulæ too, and by induction these atomic subformulæ involve at most $(m + 1) + k = m + (k + 1)$ variables, as required.

Now we prove the second statement. The rank 0 case follows because there are only finitely many terms of height at most n in the given set of m variables. Now assume the lemma is true for rank k sentences (and for all m), and consider a formula $\Phi(x_0, \dots, x_{m-1})$ of rank $k + 1$. Now $\Phi(x_0, \dots, x_{m-1})$ is a Boolean combination of formulæ of the form $\exists x_m \Psi(x_0, \dots, x_{m-1}, x_m)$, where $\Psi(x_0, \dots, x_{m-1}, x_m)$ is of rank k, for which the maximum height of any term in $\Psi(x_0, \dots, x_{m-1}, x_m)$ is of height at most n. By the induction hypothesis, there are only finitely many formulæ of the form $\Psi(x_0, \dots, x_{m-1}, x_m)$, up to logical equivalence. Hence, up to logical equivalence there are only finitely many Boolean combinations of such formulæ, which establishes the rank $k + 1$ case of the lemma (for any m). □

The next lemma is usually called Birkhoff's Finite Basis Theorem; see Theorem 4.2 of Burris and Sankappanavar [1] for example.

Birkhoff's Finite Basis Theorem 5.3. Let $\mathbf{A}$ be a finite algebra. For any number k, there is a finite set equations $\Sigma = \{s_i(x, x_1, \dots, x_k) \approx t_i(x, x_1, \dots, x_k) \mid i = 1, 2, \dots, \ell\}$ such that $\mathbf{A}$ satisfies

$$\forall x_0 \forall x_1 \dots \forall x_k \; s_i(x, x_1, \dots, x_k) \approx t_i(x, x_1, \dots, x_k), \qquad \text{for each } i,$$

and for every term $t(x, x_1, \dots, x_k)$ there is an $i \leq \ell$ such that $t(x, x_1, \dots, x_k) \approx t_i(x, x_1, \dots, x_k)$ follows by applications of equations from Σ.

Proof. Let the $u_1, \dots, u_r$ be a transversal of the $\equiv_{\mathbf{A}}$-classes of the term algebra $T(x, x_1, \dots, x_k)$. We could further assume that each u_i is minimal height in its $\equiv_{\mathbf{A}}$-class, though this will not be used.

The set Σ will consist of the following.

(1) Include $x \approx x_1$ if $|A| = 1$.
(2) For each variable $y \in \{x, x_1, \dots, x_k\}$ and $u \in \{u_1, \dots, u_r\}$ with $y \equiv_{\mathbf{A}} u$, include $y \approx u$.
(3) For each fundamental operation f (of arity n, say) and elements $u_{i_1}, \dots, u_{i_n}$ of $\{u_1, \dots, u_r\}$, let $u \in \{u_1, \dots, u_r\}$ be such that $f(u_{i_1}, \dots, u_{i_n}) \equiv_{\mathbf{A}} u$. Include the equation $f(u_{i_1}, \dots, u_{i_n}) \approx u$. (In the case that f is nullary, this definition will include any equalities between constants that hold in $\mathbf{A}$.)

We prove that every term t reduces (by applications of identities in Σ) to the right hand side of one of the equations in Σ by induction on the height k of a term t. If $k = 0$, we are done since $t \in X \subseteq \Sigma$ (by (2)).
Now assume that t is a term of height $k + 1$; that is, t is of the form $f(s_1, \dots, s_n)$ for some fundamental operation f of arity n, where s_i have height at most k. By the induction hypothesis we have $\Sigma \vdash s_i \approx u_j$ (where $i \leq n$ and $j \leq r$). Thus, after k applications of replacement, we have $\Sigma \vdash t \approx f(u_1, \dots, u_n)$. Then by (3), there exists $u \in \{u_1, \dots, u_r\}$ with $f(u_1, \dots, u_n) \approx u \in \Sigma$. Hence, using applications of equations from Σ we obtain $t \approx f(u_1, \dots, u_n) \approx u$ as required. □

Definition 5.4. Let $\mathbf{A}$ be a finite algebra and $a \in A$. Define the rank k type of the element a, denoted $\mathrm{tp}_k(\mathbf{A}, a)$, to be the set of all FO-formulæ $\varphi(x)$ of rank k for which $\mathbf{A} \models \varphi(a)$.

Lemma 5.5. *Let $\mathbf{A}$ be a finite algebra and $a \in A$. Then the rank k type $\mathrm{tp}_k(\mathbf{A}, a)$ of a is logically equivalent to a finite subset of $\mathrm{tp}_k(\mathbf{A}, a)$. The same is true for the rank type $\mathrm{tp}_k(\mathbf{A})$.*

Proof. We prove only the first statement, as the second statement is proved by a slight simplification of the first argument. We use the set Σ of equations identified in Birkhoff's Finite Basis Theorem. The set $\mathrm{tp}_k(\mathbf{A}, a)$ contains the formulæ

$$\forall x_1 \dots \forall x_k \; s_i(x, x_1, \dots, x_k) \approx t_i(x, x_1, \dots, x_k),$$

for each $i = 1, \dots, \ell$. Now let $\Psi(x)$ be a formula in $\mathrm{tp}_k(\mathbf{A}, a)$, and let

$$s(x, x_1, \dots, x_m) \approx t(x, x_1, \dots, x_m)$$

be an atomic subformula of $\Psi(x)$. Now, by the first statement in Lemma 5.2, we have that $s(x, x_1, \dots, x_m) \approx t(x, x_1, \dots, x_m)$ involves at most $k+1$ distinct variables. Hence Birkhoff's Finite Basis Theorem shows that there are $i, j \leq \ell$ such that the formula $s(x, x_1, \dots, x_m) \approx t(x, x_1, \dots, x_m)$ is equivalent to $t_i(x_{i_0}, \dots, x_{i_k}) \approx t_j(x_{i_0}, \dots, x_{i_k})$, for some subset $\{x_{i_0}, \dots, x_{i_k}\} \subseteq \{x, x_1, \dots, x_m\}$. Thus in the presence of Σ, the formula $\Psi(x)$ is equivalent to one in which all terms have height at most the maximum height of the terms $t_1, \dots, t_\ell$ (which in fact will be at most $|A|^{|A|^{k+1}}$). Thus, by the second part of Lemma 5.2, the set $\mathrm{tp}_k(\mathbf{A}, a)$ is logically equivalent to the finite subset consisting of Σ along with all members of $\mathrm{tp}_k(\mathbf{A}, a)$ whose terms are of height at most n. □

The following is similar to Theorem 3.15 of Libkin [2], but adjusted to the algebraic setting.

Lemma 5.6. *For any finite algebra* $\mathbf{A}$ *and element* $a \in A$ *there is a rank* k *formula* $\alpha_k(x)$ *such that* $\mathbf{A} \models \alpha_k(a)$ *and any similar algebra* $\mathbf{B}$ *with* $b \in B$ *satisfying* $\alpha_k(b)$ *has* $\mathrm{tp}_k(\mathbf{B}, b) = \mathrm{tp}_k(\mathbf{A}, a)$.

Proof. The rank k formula $\alpha_k(x)$ is just the conjunction of the finite subset of $\mathrm{tp}_k(\mathbf{A}, a)$ guaranteed by Lemma 5.5. Now assume that $\mathbf{B} \models \alpha_k(b)$. As $\alpha_k(x)$ is logically equivalent to $\mathrm{tp}_k(\mathbf{A}, a)$, it follows that each formula $\Phi(x) \in \mathrm{tp}_k(\mathbf{A}, a)$ has $\mathbf{B} \models \Phi(b)$. Now assume that $\Phi(x)$ is a rank k formula in $\mathrm{tp}_k(\mathbf{B}, b)$; we show that $\Phi(x) \in \mathrm{tp}_k(\mathbf{A}, a)$ as well. Indeed, otherwise $\mathbf{A} \models \neg\Phi(a)$, so that $\neg\Phi(x) \in \mathrm{tp}_k(\mathbf{A}, a)$. But then we would have $\mathbf{B} \models \neg\Phi(b)\ \&\ \Phi(b)$, a contradiction. $\square$

We shall prove the equivalence of (1) and (2) in the Theorem 4.1, as well as a new condition, the back-and-forth equivalence.

Definition 5.7. We inductively define the back-and-forth relations $\simeq_k$ on finite $\mathcal{F}$-structures $\mathbf{A}$ and $\mathbf{B}$ as follows.

- $\mathbf{A} \simeq_0 \mathbf{B}$ iff $\mathbf{A} \equiv_0 \mathbf{B}$; that is, $\mathbf{A}$ and $\mathbf{B}$ satisfy the same atomic sentences.
- $\mathbf{A} \simeq_{k+1} \mathbf{B}$ iff the following two conditions hold:
  forth: for every $a \in \mathbf{A}$, there exists $b \in \mathbf{B}$ such that $(\mathbf{A}, a) \simeq_k (\mathbf{B}, b)$.
  back: for every $b \in \mathbf{B}$, there exists $a \in \mathbf{A}$ such that $(\mathbf{A}, a) \simeq_k (\mathbf{B}, b)$.

We now prove the following extension of Theorem 4.1.

Theorem 5.8. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be two finite* $\mathcal{F}$*-structures. Then the following are equivalent:*

(1) $\mathbf{A}$ *and* $\mathbf{B}$ *agree on* $\mathrm{FO}[k]$;
(2) $\mathbf{A} \equiv_k \mathbf{B}$;
(3) $\mathbf{A} \simeq_k \mathbf{B}$.

Proof. By induction on k. The $k = 0$ case is obvious. First we prove the equivalence of (2) and (3) followed by the equivalence of (1) and (3).
((3) $\implies$ (2)). Assume that $\mathbf{A} \simeq_{k+1} \mathbf{B}$; that is the conditions **forth** and **back** above hold. We must show that $\mathbf{A} \equiv_{k+1} \mathbf{B}$. Without loss of generality assume Spoiler plays $a \in \mathbf{A}$ as his first move. Then by **forth**, we can find $b \in \mathbf{B}$ with $(\mathbf{A}, a) \simeq_k (\mathbf{B}, b)$. Thus, by the hypothesis, we have $(\mathbf{A}, a) \equiv_k (\mathbf{B}, b)$; that is, Duplicator can continue to play for another k moves, and thus wins the $k+1$-round game. Hence $\mathbf{A} \equiv_{k+1} \mathbf{B}$ as required. The other direction is similar.

((1) $\implies$ (3)). Assume $\mathbf{A}$ and $\mathbf{B}$ agree on $\mathrm{FO}[k+1]$; that is, for every sentence Ψ of $\mathrm{qr}^+(\Psi) = k+1$ we have $\mathbf{A} \models \Psi$ iff $\mathbf{B} \models \Psi$. We show that $\mathbf{A} \simeq_{k+1} \mathbf{B}$. We shall prove the **forth** case only (the **back** case is identical). Choose $a \in \mathbf{A}$. Then, by Lemma 5.6, its rank-k type $\mathrm{tp}_k(\mathbf{A}, a)$ is logically equivalent to the rank-k formula $\alpha_k(x)$ and so $\mathbf{A} \models \alpha_k(a)$. Thus, $\mathbf{A} \models \exists x \alpha_k(x)$ and since $\alpha_k(x)$ is a formula of quantifier rank k; this is a sentence of quantifier rank $k+1$. Hence $\mathbf{B} \models \exists x \alpha_k(x)$ by assumption. Now let b be the witness for the existential quantifier, then by Lemma 5.6 we have $\mathrm{tp}_k(\mathbf{B}, b) = \mathrm{tp}_k(\mathbf{A}, a)$. Hence for every sentence Φ of $\mathrm{qr}^+(\Phi) = k+1$ we have $(\mathbf{A}, a) \models \Phi$ iff $(\mathbf{B}, b) \models \Phi$; that is, $(\mathbf{A}, a)$ and $(\mathbf{B}, b)$ agree on $\mathrm{FO}[k]$. Then the hypothesis implies $(\mathbf{A}, a) \simeq_k (\mathbf{B}, b)$; and by definition this is $\mathbf{A} \simeq_{k+1} \mathbf{B}$.

((3) $\implies$ (1)). In the other direction, we assume $\mathbf{A} \simeq_{k+1} \mathbf{B}$ and we show $\mathbf{A}$ and $\mathbf{B}$ agree on FO$[k+1]$. Since every FO$[k+1]$ sentence is a Boolean combination (that is, constructed using connectives $\wedge$, $\vee$ and $\neg$ only) of $\exists x\varphi(x)$, it suffices to prove the result for sentences of the form $\exists x\varphi(x)$. Assume that $\mathbf{A} \models \exists x\varphi(x)$. Thus, $\mathbf{A} \models \varphi(a)$ for some $a \in \mathbf{A}$. By **forth**, we can find $b \in \mathbf{B}$ such that $(\mathbf{A}, a) \simeq_k (\mathbf{B}, b)$. Hence $(\mathbf{A}, a)$ and $(\mathbf{B}, b)$ agree on FO$[k]$ by the hypothesis. Thus $\mathbf{B} \models \varphi(b)$ and so $\mathbf{B} \models \exists x\varphi(x)$. The converse ($\mathbf{B} \models \exists x\varphi(x) \implies \mathbf{A} \models \exists x\varphi(x)$) is essentially identical which completes the proof. □

Finally, we sketch a proof that Corollary 4.2 is complete for properties defining uniformly locally finite classes of finite algebras (that is, classes of finite algebras for which the size of n-generated subalgebras is bounded by some positive integer). In the following theorem, it is implicit that all algebras are of the same finite signature.

Theorem 5.9. *Let $\mathcal{P}$ denote a uniformly locally finite class of finite algebras. Then $\mathcal{P}$ is definable amongst finite algebras by a first order sentence iff there exists a number k such that for every two finite algebras $\mathbf{A}$ and $\mathbf{B}$, if $\mathbf{A} \in \mathcal{P}$ and $\mathbf{A} \equiv_k \mathbf{B}$, then $\mathbf{B} \in \mathcal{P}$.*

Proof. Let $\mathbf{A}$ and $\mathbf{B}$ be finite algebras and assume that $\mathcal{P}$ is definable by a FO sentence Φ. Let $k := \mathrm{qr}^+(\Phi)$. If $\mathbf{A} \in \mathcal{P}$, then $\mathbf{A} \models \Phi$, and hence for $\mathbf{B}$ with $\mathbf{A} \equiv_k \mathbf{B}$ we have $\mathbf{B} \models \Phi$. Thus $\mathbf{B} \in \mathcal{P}$.

The converse direction requires a slightly more detailed version of Lemma 5.5 and we do not give full details. Assume that $\mathbf{A} \in \mathcal{P}$ and $\mathbf{A} \equiv_k \mathbf{B}$ imply $\mathbf{B} \in \mathcal{P}$. So in particular, any two finite algebras with the same rank-k type agree on membership in $\mathcal{P}$. Now, as $\mathcal{P}$ is uniformly locally finite and the signature is finite, there are only finitely many k-generated subalgebras of members of $\mathcal{P}$, up to isomorphism. This enables one to show that the distinct rank k types of members of $\mathcal{P}$ are finite in number: essentially, the equations Σ guaranteed by Birkhoff's Finite Basis Theorem can be chosen uniformly across all $\mathbf{A} \in \mathcal{P}$ (basically, the equational class generated by $\mathcal{P}$ will be locally finite: whence the k-generated free algebra will be finite). It is these equations (in the proof of Lemma 5.5) that guarantee the equivalence of $\mathrm{tp}(\mathbf{A})$ with some finite subset; moreover, the size of this subset depends only on k and the size of the equations in Σ.

Let Φ denote the disjunction of all the rank k-types of algebras in $\mathcal{P}$. Certainly every model of $\mathcal{P}$ satisfies Φ. But also, a model $\mathbf{B}$ of Φ shares the same rank k type as some algebra $\mathbf{A} \in \mathcal{P}$; whence such a $\mathbf{B}$ is in $\mathcal{P}$. Hence Φ defines $\mathcal{P}$ amongst finite algebras. □

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